A QUILLEN STRATIFICATION THEOREM FOR MODULES

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Let $G$ be a finite group and $k$ a fixed algebraically closed field of characteristic $p > 0$. If $p$ is odd, let $H_G$ be the subring of $H^*(G, k)$ consisting of elements of even degree; take $H_G = H^*(G, k)$ if $p = 2$. $H_G$ is a finitely generated commutative $k$-algebra, and we let $V_G$ denote its associated affine variety $\text{Max } H_G$. If $M$ is any finitely generated $kG$-module, the cohomology variety $V_G(M)$ of $M$ may be defined as the support in $V_G$ of the $H_G$-module $H^*(G, M)$ if $G$ is a $p$-group, and in general as the largest support of $H^*(G, L \otimes M)$ where $L$ is any $kG$-module. A module $L$ with each irreducible $kG$-module as a direct summand will do [3].

D. Quillen [9, 10] proved a number of beautiful results relating $V_G$ to the varieties $V_E$ associated with the elementary abelian $p$-subgroups $E$ of $G$, culminating in his stratification theorem. This theorem gives a piecewise description of $V_G$ in terms of the subgroups $E$ and their normalizers in $G$. Some of Quillen’s results have been extended to the variety $V_G(M)$ associated with a $kG$-module $M$ [1, 4, 5, 6, 7, 8], and the work of Alperin and Evens [2] and Avrunin [3] showed that there was at least a surjection $\Pi_E V_E(M) \to V_G(M)$. However, the stratification theorem for $V_G(M)$ remained elusive, since one still needed to know that a point in $V_G(M)$ in the image of a given $V_E$ was in fact in the image of $V_E(M)$.

We announce here a proof of the stratification theorem for $V_G(M)$, as well as a proof of a conjecture of J. Carlson regarding $V_E(M)$ for $E$ an elementary abelian $p$-subgroup. We are also able to generalize several of Quillen’s other results to the module case.

For $H < G$, let $t_{G,H} : V_H \to V_G$ be the transfer map induced by restriction on the cohomology rings. For an elementary abelian $p$-subgroup $E$, let $V_E^+ = V_E \setminus \bigcup_{F < E} t_{E,F} V_F$ and let $V_E^+(M) = V_E^+ \cap V_E(M)$. Then put $V_{G,E}^+(M) = t_{G,E} V_E^+ \cap V_G(M)$. We have the following stratification theorem.

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Theorem. The variety $V^+_G(M)$ is the disjoint union of its subvarieties $V^+_{G,E}(M)$, where $E$ ranges over a set of representatives for the conjugacy classes of elementary abelian $p$-subgroups of $G$. Moreover, each of the varieties $V^+_{G,E}(M)$ and $V^+_E(M)$ is affine, the group $N_G(E)/C_G(E)$ acts freely on $V^+_E(M)$, and $t_{G,E}$ induces a bijective finite morphism

$$V^+_E(M)/(N_G(E)/C_G(E)) \to V^+_{G,E}(M).$$

To establish the theorem, we first prove Carlson's conjecture equating $V^+_E(M)$ for $E$ elementary abelian with a variety, the "rank variety", defined more directly in terms of the action of $E$ on $M$. Let $L$ be a $k$-subspace of $kE$ with $J = L \oplus J^2$, where $J$ is the kernel of the augmentation map. Then $kE$ is the restricted enveloping algebra $u(L)$ of $L$, regarding $L$ as a commutative restricted Lie algebra with trivial $p$th power. $H_L$, $V_L$, and $V_L(M)$ are defined just as in the group case, and one sees easily that $H_L = H_E$, $V_L = V_E$, and $V_L(M) = V^+_E(M)$. There is also a natural identification $L = V^+_L = V^+_E$. (For $p = 2$ this comes from the isomorphism $H^1(L, k) \cong L^*$.). We define the rank variety $V^*_L(M)$ to be the union of all 1-dimensional $k$-subspaces $S$ of $L$ (automatically restricted Lie subalgebras) for which $M|_S$ is not projective. Carlson, whose original definition [5, 6] of the rank variety was in terms of "shifted subgroups" of $kE$ whose group algebras are generated by the subspaces of $L$, showed that $V^*_L(M)$ is a variety of dimension equal to that of $V^+_E(M)$ (see also Kroll [8]), and that, under the natural identification, $V^*_L(M) \subseteq V^+_E(M)$. He then conjectured

Theorem (Carlson's conjecture). $V^*_L(M) \cong V^+_E(M)$.

If $T$ is a subalgebra of $L$ and $t_{L,T} : V_T \to V_L$ is the map induced by restriction on cohomology, we have $T \cong t_{L,T}V_T$ in the identification $L \cong V_L$. To prove Carlson's conjecture, let $S$ be a 1-dimensional subalgebra of $L$ with $S = t_{L,S}V_S \subseteq V_L(M)$. We have to show $M|_S$ is not projective. If $M|_S$ is projective, a spectral sequence argument gives $H^*(L/S, M^S) \cong H^*(L, M)$, where the isomorphism is inflation followed by the map on cohomology induced by the inclusion $M^S \subseteq M$. It follows that $H^*(L, M)$ is a finitely generated $H_{L/S}$-module. But the inflation of the ideal of all elements of positive degree in $H_{L/S}$ is contained in the ideal $P$ of $S = t_{L,S}V_S$ in $H_L$, so $H^*(L, M)/P \cdot H^*(L, M)$ is a finite-dimensional $k$-space. By Nakayama's lemma, one then sees that the support $V_L(M)$ of $H^*(L, M)$ in $V_L$ contains only finitely many points of $S$, which is a contradiction.

As a corollary of Carlson's conjecture, we obtain the following result in the special case that $G$ is an elementary abelian $p$-group.

Theorem. Let $G$ be a finite group and $H$ a subgroup of $G$. If $M$ is a finitely generated $kG$-module, then $V_H(M) = t_{G,H}^{-1}V_G(M)$.
To prove this theorem in general, we recall from [3] that $V_H(M) \subseteq t_{G,H}^{-1}V_G(M)$. Suppose $v \in t_{G,H}^{-1}V_G(M)$. By [2] or [3] we can choose an elementary subgroup $E$ and an $s \in V_E(M)$ with $t_{G,E}(S) = t_{G,H}(v)$. Quillen's stratification theorem says that there exists an elementary subgroup $E' \leq H$ and $s' \in V_{E'}^+$ with $t_{H,E}(s') = v$, and that some conjugate of $s'$ maps to $s$ under the appropriate transfer map. By the corollary to Carlson's conjecture, we have $s' \in V_{E'}^+(M)$ and this implies [3] that $v \in V_H(M)$. Thus $t_{G,H}^{-1}V_G(M) \subseteq V_H(M)$, and the theorem is proved.

The stratification theorem for modules now follows from Quillen's original theorem and this result.

For any subgroup $H$ of $G$, let $r_H(M)$ denote the radical ideal in $H_H$ defining $V_H(M)$ as a subvariety of $V_H$. (If $H$ is a $p$-group, $r_H(M)$ is the radical of the annihilator of $H^*(H, M)$ in $H_H$. A similar interpretation can be given in general; see [3].) Using the stratification theorem above, we can generalize a "glueing theorem" of Quillen's to obtain

**Theorem.** Let $F$ be a family of elementary abelian $p$-subgroups of $G$ which is closed under conjugation and taking subgroups. Suppose, for each $E \in F$, we have an element $\gamma_E \in H_E$ such that, for any $E' \in F$ and any restriction or conjugation map $H_E \to H_{E'}$, $\gamma_E$ is sent to an element of the coset $\gamma_{E'} + r_{E'}(M)$. Then there exists an element $\gamma \in H_G$ and a power $q$ of $p$ such that, for each $E \in F$,

$$\gamma|_E \equiv \gamma_E^q \mod r_E(M).$$

Applying the result $V_H(M) = t_{G,H}^{-1}V_G(M)$ for $H \leq G$, obtained in the course of proving the stratification, to the diagonal embedding $G \to G \times G$, we get the following tensor product theorem, due to Carlson [6] in the case of elementary abelian $p$-groups.

**Theorem.** Let $M$ and $N$ be finitely generated $kG$-modules. Then

$$V_G(M \otimes_k N) = V_G(M) \cap V_G(N).$$

Further details of the proofs and additional results will appear elsewhere.

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