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A QUILLEN STRATIFICATION THEOREM FOR MODULES

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Let G be a finite group and k a fixed algebraically closed field of characteristic $p > 0$. If p is odd, let H_G be the subring of $H^*(G, k)$ consisting of elements of even degree; take $H_G = H^*(G, k)$ if $p = 2$. H_G is a finitely generated commutative k-algebra, and we let V_G denote its associated affine variety Max H_G. If M is any finitely generated kG-module, the *cohomology variety* $V_G(M)$ of M may be defined as the support in V_G of the H_G-module $H^*(G, M)$ if G is a p-group, and in general as the largest support of $H^*(G, L \otimes M)$ where L is any kG-module. A module L with each irreducible kG-module as a direct summand will do [3].

D. Quillen [9, 10] proved a number of beautiful results relating V_G to the varieties V_E associated with the elementary abelian p-subgroups E of G, culminating in his stratification theorem. This theorem gives a piecewise description of V_G in terms of the subgroups E and their normalizers in G. Some of Quillen's results have been extended to the variety $V_G(M)$ associated with a kG-module M [1, 4, 5, 6, 7, 8], and the work of Alperin and Evens [2] and Avrunin [3] showed that there was at least a surjection $\amalg_E V_E(M) \longrightarrow V_G(M)$. However, the stratification theorem for $V_G(M)$ remained elusive, since one still needed to know that a point in $V_G(M)$ in the image of a given V_E was in fact in the image of $V_E(M)$.

We announce here a proof of the stratification theorem for $V_G(M)$, as well as a proof of a conjecture of J. Carlson regarding $V_E(M)$ for E an elementary abelian p-subgroup. We are also able to generalize several of Quillen's other results to the module case.

For $H < G$, let $t_{G,H}: V_H \longrightarrow V_G$ be the transfer map induced by restriction on the cohomology rings. For an elementary abelian p-subgroup E, let $V_E^+ = V_E \backslash \bigcup_{F<E} t_{E,F} V_F$ and let $V_E^+(M) = V_E^+ \cap V_E(M)$. Then put $V_{G,E}^+(M) = t_{G,E} V_E^+ \cap V_G(M)$. We have the following stratification theorem.

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THEOREM. *The variety $V_G(M)$ is the disjoint union of its subvarieties $V^+_{G,E}(M)$, where E ranges over a set of representatives for the conjugacy classes of elementary abelian p-subgroups of G. Moreover, each of the varieties $V^+_{G,E}(M)$ and $V^+_E(M)$ is affine, the group $N_G(E)/C_G(E)$ acts freely on $V^+_E(M)$, and $t_{G,E}$ induces a bijective finite morphism*

$$V^+_E(M)/(N_G(E)/C_G(E)) \longrightarrow V^+_{G,E}(M).$$

To establish the theorem, we first prove Carlson's conjecture equating $V_E(M)$ for E elementary abelian with a variety, the "rank variety", defined more directly in terms of the action of E on M. Let L be a k-subspace of kE with $J = L \oplus J^2$, where J is the kernel of the augmentation map. Then kE is the restricted enveloping algebra $u(L)$ of L, regarding L as a commutative restricted Lie algebra with trivial pth power. H_L, V_L, and $V_L(M)$ are defined just as in the group case, and one sees easily that $H_L = H_E$, $V_L = V_E$, and $V_L(M) = V_E(M)$. There is also a natural identification $L \cong V_L = V_E$. (For $p = 2$ this comes from the isomorphism $H^1(L, k) \cong L^*$.) We define the *rank variety* $V^r_L(M)$ to be the union of all 1-dimensional k-subspaces S of L (automatically restricted Lie subalgebras) for which $M|_S$ is not projective. Carlson, whose original definition **[5, 6]** of the rank variety was in terms of "shifted subgroups" of kE whose group algebras are generated by the subspaces of L, showed that $V^r_L(M)$ is a variety of dimension equal to that of $V_E(M)$ (see also Kroll [8]), and that, under the natural identification, $V^r_L(M) \subseteq V_E(M)$. He then conjectured

THEOREM (CARLSON'S CONJECTURE). $V^r_L(M) \cong V_E(M)$.

If T is a subalgebra of L and $t_{L,T}\colon V_T \longrightarrow V_L$ is the map induced by restriction on cohomology, we have $T \cong t_{L,T}V_T$ in the identification $L \cong V_L$. To prove Carlson's conjecture, let S be a 1-dimensional subalgebra of L with $S = t_{L,S}V_S \subseteq V_L(M)$. We have to show $M|_S$ is not projective. If $M|_S$ is projective, a spectral sequence argument gives $H^*(L/S, M^S) \xrightarrow{\sim} H^*(L, M)$, where the isomorphism is inflation followed by the map on cohomology induced by the inclusion $M^S \subset M$. It follows that $H^*(L, M)$ is a finitely generated $H_{L/S}$-module. But the inflation of the ideal of all elements of positive degree in $H_{L/S}$ is contained in the ideal $\mathcal{P}$ of $S = t_{L,S}V_S$ in H_L, so $H^*(L, M)/\mathcal{P} \cdot H^*(L, M)$ is a finite-dimensional k-space. By Nakayama's lemma, one then sees that the support $V_L(M)$ of $H^*(L, M)$ in V_L contains only finitely many points of S, which is a contradiction.

As a corollary of Carlson's conjecture, we obtain the following result in the special case that G is an elementary abelian p-group.

THEOREM. *Let G be a finite group and H a subgroup of G. If M is a finitely generated kG-module, then $V_H(M) = t^{-1}_{G,H}V_G(M)$.*

To prove this theorem in general, we recall from [3] that $V_H(M) \subseteq t_{G,H}^{-1} V_G(M)$. Suppose $v \in t_{G,H}^{-1} V_G(M)$. By [2] or [3] we can choose an elementary subgroup E and an $s \in V_E(M)$ with $t_{G,E}(S) = t_{G,H}(v)$. Quillen's stratification theorem says that there exists an elementary subgroup $E' \leqslant H$ and $s' \in V_{E'}^+$ with $t_{H,E'}(s') = v$, and that some conjugate of s' maps to s under the appropriate transfer map. By the corollary to Carlson's conjecture, we have $s' \in V_{E'}(M)$ and this implies [3] that $v \in V_H(M)$. Thus $t_{G,H}^{-1} V_G(M) \subseteq V_H(M)$, and the theorem is proved.

The stratification theorem for modules now follows from Quillen's original theorem and this result.

For any subgroup H of G, let $r_H(M)$ denote the radical ideal in H_H defining $V_H(M)$ as a subvariety of V_H. (If H is a p-group, $r_H(M)$ is the radical of the annihilator of $H^*(H, M)$ in H_H. A similar interpretation can be given in general; see [3].) Using the stratification theorem above, we can generalize a "glueing theorem" of Quillen's to obtain

THEOREM. *Let $\mathcal{F}$ be a family of elementary abelian p-subgroups of G which is closed under conjugation and taking subgroups. Suppose, for each $E \in \mathcal{F}$, we have an element $\gamma_E \in H_E$ such that, for any $E' \in \mathcal{F}$ and any restriction or conjugation map $H_E \longrightarrow H_{E'}$, γ_E is sent to an element of the coset $\gamma_{E'} + r_{E'}(M)$. Then there exists an element $\gamma \in H_G$ and a power q of p such that, for each $E \in \mathcal{F}$,*

$$\gamma|_E \equiv \gamma_E^q \pmod{r_E(M)}.$$

Applying the result $V_H(M) = t_{G,H}^{-1} V_G(M)$ for $H \leqslant G$, obtained in the course of proving the stratification, to the diagonal embedding $G \longrightarrow G \times G$, we get the following tensor product theorem, due to Carlson [6] in the case of elementary abelian p-groups.

THEOREM. *Let M and N be finitely generated kG-modules. Then*

$$V_G(M \otimes_k N) = V_G(M) \cap V_G(N).$$

Further details of the proofs and additional results will appear elsewhere.

It is a pleasure to acknowledge the contributions of Jon Carlson and Jon Alperin to this research. The notion of a tensor product theorem was Carlson's very original idea. Using this, Alperin was able to give a reduction of the stratification problem, which we followed in our original proof. We would also like to thank Peter Donovan for sharing with us some of his ideas on cohomology rings.

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