• Exercise 9.14: Prove Theorem 9.11. (This theorem says that if $K \subseteq L$ is a field extension and $\gamma \in L$ is algebraic over $K$ of degree $n$, with $f(x) \in K[x]$ a polynomial of degree $n$ with $f(\gamma) = 0$, and $g(x) \in K[x]$ is any polynomial with $g(\gamma) = 0$, then $f(x)$ divides $g(x)$.)

We know that there exist $q(x), r(x) \in K[x]$ such that $g(x) = f(x)q(x) + r(x)$ and $\deg r(x) < \deg f(x)$. Since $g(\gamma) = f(\gamma) = 0$, we must have $r(\gamma) = 0$ as well. But $\gamma$ is algebraic of degree $n$, so no nonzero polynomial in $K[x]$ of degree less than $n$ has $\gamma$ as a root, and $r(x)$ has degree less than $n$. So we must have $r(x) = 0$. Then $g(x) = f(x)q(x)$, so $f(x)$ divides $g(x)$. \qed

• Exercise 10.1: Let us review how the quadratic formula is obtained.

1. For a real number $a$, verify that $(x + a)^2 = x^2 + 2xa + a^2$.

   We have
   \[ (x + a)^2 = x^2 + xa + ax + a^2 = x^2 + 2xa + a^2. \]

2. For a real number $b$, conclude that the polynomial $x^2 + bx + \frac{b^2}{4}$ is the square of a degree-one polynomial.

   Taking $a = \frac{b}{2}$ and applying the result of part 1, we see that $x^2 + bx + \frac{b^2}{4} = (x + \frac{b}{2})^2$.

3. For real numbers $b$ and $c$, rewrite $x^2 + bx + c$ by adding and subtracting $\frac{b^2}{4}$ and find that solving the equation $x^2 + bx + c$ is equivalent to solving an equation of the form $(x + \frac{b}{2})^2 = \frac{d}{4}$ for a suitable number $d$. Write out the number $d$ explicitly in terms of $b$ and $c$. This number is called the discriminant of the polynomial $x^2 + bx + c$. It will be important in what follows.

   We have
   \[ x^2 + bx + c = x^2 + bx + \frac{b^2}{4} + c - \frac{b^2}{4} \]
   \[ = (x + \frac{b}{2})^2 + \frac{4c - b^2}{4}. \]

   So a $r \in \mathbb{R}$ is a solution to $x^2 + bx + c$ if and only if it’s a solution to $(x + \frac{b}{2})^2 + \frac{4c-b^2}{4} = 0$ or, equivalently, to $(x + \frac{b}{2})^2 = \frac{b^2-4c}{4}$. We can take $d = b^2 - 4c$ to get the from the question asks for.
4. Deduce that if \( d = 0 \), then \( x^2 + bx + c \) factors as \((x + \frac{b}{2})^2\), and the one solution to \( x^2 + bx + c = 0 \) is \( x = -\frac{b}{2} \).

If \( d = 0 \), then we are interested in solutions to \((x + \frac{b}{2})^2 = 0\). If \( r \) is a solution, we must have \( r = -\frac{b}{2} \)(since \( \mathbb{R} \) is a field and hence an integral domain, a product is 0 if and only if at least one of the factors is 0).

5. Deduce that if \( d \) is negative, then there is no solution in \( \mathbb{R} \) to the equation \( x^2 + bx + c = 0 \), and \( x^2 + bx + c \) is irreducible in \( \mathbb{R}[x] \).

If \( d \) is negative, we are looking for solutions to the equation \((x + \frac{b}{2})^2 = \frac{d}{4}\). But for all \( r \in \mathbb{R} \), \((r + \frac{b}{2})^2\) is the square of a real number and therefore nonnegative. Hence there are no solutions in \( \mathbb{R} \) to the equation \((x + \frac{b}{2})^2 = \frac{d}{4}\). By part 3, that says there are no real solutions to \( x^2 + bx + c = 0 \).

6. Deduce that if \( d \) is positive, then there are two real solutions to \( x^2 + bx + c = 0 \). Write out explicitly what these solutions are in terms of \( b \) and \( c \).

If \( d \) is positive, we see that the solutions to the equation \((x + \frac{b}{2})^2 = \frac{d}{4}\) are exactly the solutions to \( x + \frac{b}{2} = \pm\sqrt{\frac{d}{4}} \).

These are \( r_1 = -\frac{b}{2} + \frac{\sqrt{d}}{2} \) and \( r_2 = -\frac{b}{2} - \frac{\sqrt{d}}{2} \), where \( d = b^2 - 4c \).

- Exercise 10.2: Suppose \( b \) and \( c \) are real numbers. Let \( r_1 \) and \( r_2 \) be the two real or complex roots of the polynomial \( x^2 + bx + c \) [counting multiplicity, so that if there is only one root, it counts twice—we’ll cover this later].

1. Observe that if \( r_1 \) and \( r_2 \) are real and distinct, then \((r_1 - r_2)^2 > 0\).

   If \( r_1 \) and \( r_2 \) are real and distinct, \( r_1 - r_2 \) is a nonzero real number so its square is positive.

2. Observe that if \( r_1 = r_2 \), then \((r_1 - r_2)^2 = 0\).

   You don’t really have to say anything here; \( 0^2 = 0 \).

3. The only remaining possibility is that \( r_1 \) and \( r_2 \) are nonreal, complex numbers that are complex conjugates of each other. In this case we can write \( r_1 \) as \( s + ti \) and \( r_2 \) as \( s - ti \) for some real numbers \( s, t \) with \( t \neq 0 \). (Why can we assume \( t \neq 0 \)?) Calculate \((r_1 - r_2)^2 \) and show that it is a negative real number.

   If \( t = 0 \), then \( r_1 \) and \( r_2 \) are real and equal to each other; a case that’s already been covered. So if \( r_1, r_2 \) are not real, we must have \( t \neq 0 \).

   In this case \((r_1 - r_2)^2 = (2ti)^2 = 4t^2i^2 = -4t^2 \). Since \( t \) is real, \( 4t^2 > 0 \) and \(-4t^2 \) is negative.
4. Write $\Delta$ for $(r_1 - r_2)^2$. We have found that $\Delta$ is positive, zero, or negative depending on whether $r_1$ and $r_2$ are real and distinct, identical, or nonreal. Using these facts, show that the converse holds as well:

(a) If $\Delta > 0$, there are two distinct real roots;
(b) If $\Delta = 0$, there is a real root with multiplicity 2;
(c) If $\Delta < 0$, there are two distinct complex conjugate roots.

If $\Delta > 0$, then $(r_1 - r_2)^2$ is positive. This means that $r_1 - r_2 = \pm \sqrt{\Delta}$. We know the only possibilities for $r_1, r_2$ are distinct real numbers, 0, or a pair of nonreal complex numbers that are complex conjugates. So $r_1 - r_2$ is clearly not 0, and the difference of two nonreal complex numbers that are conjugate is a (nonzero) purely imaginary number of the form $ti$ whose square must be negative. So it must be the case that $r_1, r_2$ are distinct real numbers.

If $\Delta = 0$, we must have $r_1 = r_2$, and we have seen that this means the root is real.

If $\Delta < 0$, then the square of $r_1 - r_2$ is a negative real number. This means that $r_1 - r_2$ is a nonreal complex number of the form $ti$, and hence $r_1, r_2$ are nonreal complex conjugates.

• **Exercise 10.3:** Continue with the notation of Exercise 10.2, with the quadratic polynomial $x^2 + bx + c$ factoring as $(x - r_1)(x - r_2)$. Use this to express $b$ and $c$ in terms of $r_1$ and $r_2$. Deduce that $(r_1 - r_2)^2 = b^2 - 4c$. Thus $\Delta$ and $b^2 - 4c$ coincide.

If $r_1, r_2$ are roots of the polynomial $x^2 - bx + c$ in $\mathbb{C}$ (possibly in $\mathbb{R} \subseteq \mathbb{C}$, but definitely in $\mathbb{C}$), then Theorem 10.2 tells us that the polynomial $s^2 - bx + c$ factors in $\mathbb{C}[x]$ as $(x-r_1)(x-r_2)$. Since $(x-r_1)(x-r_2) = x^2 - (r_1 + r_2)x + r_1r_2$, we see that $b = -(r_1 + r_2)$ and $c = r_1r_2$. Then

\[
\begin{align*}
    b^2 - 4c &= (-r_1 - r_2)^2 - 4r_1r_2 \\
    &= r_1^2 + 2r_1r_2 + r_2^2 - 4r_1r_2 \\
    &= r_1^2 - 2r_1r_2 + r_2^2 = (r_1 - r_2)^2.
\end{align*}
\]

• **Exercise 10.5:** Let $f(x)$ be a cubic polynomial of the form $x^3 + ax^2 + bx + c$ with real coefficients.

1. Review an argument from calculus that allows you to deduce that $f(x) \to \infty$ as $x \to \infty$. Similarly, $f(x) \to -\infty$ as $x \to -\infty$.

2. Read a discussion of the intermediate value theorem in a calculus book. This is the foundational result real numbers, on which all of calculus depends. Without it, calculus would not work. It basically says that if the graph of a continuous function starts at some height
and ends at another, then the graph goes through every height in between.

3. Using the intermediate value theorem, deduce for our cubic polynomial \( f(x) \) that there is a real number \( r \) such that \( f(r) = 0 \).

   If \( \lim_{x \to \infty} f(x) = \infty \), then for large values of \( x \), \( f(x) > 0 \).
   Similarly, \( \lim_{x \to -\infty} f(x) = -\infty \) tells us that, for large (positive) values of \( x \), \( f(-x) < 0 \). So \( f(x) \) is positive for some values of \( x \) and negative for other values of \( x \). Since a polynomial function \( \mathbb{R} \to \mathbb{R} \) is continuous (see your calculus book; \( g(x) = x \) is easily seen to be continuous, as are constant functions, and the sum and product of continuous functions is continuous), the intermediate value theorem says there exists an \( r \) with \( f(r) = 0 \).

4. Conclude that \( f(x) \) can be factored as \( (x - r)g(x) \) for some quadratic polynomial \( g(x) \).

   This is just Theorem 9.7 and the fact that \( f(x) \) has degree 3.

5. Deduce that either \( f(x) \) factors in \( \mathbb{R}[x] \) as the product of three degree-one polynomials, or \( f(x) \) factors in \( \mathbb{R}[x] \) as the product of a degree-one polynomial and an irreducible degree-two polynomial.

   Since \( f(x) = (x - r)g(x) \), the question really has to do with how the quadratic polynomial \( g(x) \) factors.

   If \( g(x) \) is irreducible in \( \mathbb{R}[x] \) (since it’s quadratic, this is the case where it has no real roots), it can’t be nontrivially factored further in \( \mathbb{R}[x] \). But if \( g(x) \) is not irreducible, it must factor as a product of two polynomials of lower (but positive) degree. Since \( \deg g(x) = 2 \), these polynomials must both have degree 1. (See Exercise 9.13.)

6. Deduce that either \( f(x) \) has three real roots (counting multiplicities) or \( f(x) \) has one real root and two nonreal (complex) roots that are complex conjugates of each other.

   If \( f(x) \) factors in \( \mathbb{R}[x] \) as a product of 3 degree-1 polynomials, it has 3 real roots (counting multiplicities): a monic degree 1 polynomial has the form \( x - r \) and therefore has the root \( r \). Clearly the root of each degree-1 factor is a root of the product of the degree-1 factors. On the other hand, if \( f(x) \) factors in \( \mathbb{R}[x] \) as the product of a degree-1 polynomial and an irreducible degree-2 polynomial, it has only one real root (the root of the degree-1 factor). The irreducible degree-2 factor has no real roots, but it has 2 nonreal complex roots that are conjugates of each other. Those are roots of \( f(x) \) in \( \mathbb{C} \).
• Exercise 10.6: What is the reduced cubic equation that you must solve in order to solve the cubic equation \( x^3 - 3x^2 - 4x + 12 = 0 \)? [Note that “must” is too strong here; he’s asking for the reduced cubic you’d solve if you’re using Cardano’s formula.]

The change of variable we want to make is to let \( x = y - \frac{3}{4}, \) where \( a \) is the coefficient of \( x^2 \). Here, \( a = -3 \), so we have \( x = y + 1 \). This gives

\[
x^3 - 3x^2 - 4x + 12 = (y + 1)^3 - 3(y + 1)^2 - 4(y + 1) + 12
\]

\[
= (y^3 + 3y^2 + 3y + 1) - 3(y^2 + 2y + 1) - 4y - 4 + 12
\]

\[
= y^3 - 7y + 6.
\]

• Exercise 10.7: We begin with the cubic polynomial \( y^3 + py + q \). We can assume that \( p \) is nonzero, for if \( p = 0 \), the equation is \( y^3 = -q \), and the solution is easily obtained as the cube root of \(-q\). Viète’s idea is to introduce a new variable satisfying \( y = z - \frac{p}{3z} \).

1. Substitute \( z - \frac{p}{3z} \) for \( y \) in the equation \( y^3 + py + q = 0 \), expand the cubed term, simplify, and obtain the following equation in \( z \):

\[
z^3 - \frac{p^3}{27z^3} + q = 0.
\]

Making the substitution, we have

\[
y^3 + py + q = \left( z - \frac{p}{3z} \right)^3 + p \left( z - \frac{p}{3z} \right) + q
\]

\[
= z^3 - 3z^2 \cdot \frac{p}{3z} + 3z \left( \frac{p}{3z} \right)^2 - \left( \frac{p}{3z} \right)^3 + pz - \left( \frac{p^3}{3z} \right) + q
\]

\[
= z^3 - 3zp + \frac{p^2}{3z} - \frac{p^3}{27z^3} + pz - \frac{p^2}{3z} + q
\]

\[
= z^3 - \frac{p^3}{27z^3} + q.
\]

So the change of variable transforms the equation \( y^3 + py + q = 0 \) to the equation \( z^3 - \frac{p^3}{27z^3} + q = 0 \).

2. Multiply by \( z^3 \) to clear the variable in the denominator and obtain \( z^6 + qz^3 - \frac{p^3}{27} = 0 \).

This is clear.

3. Observe that this is a quadratic equation in \( z^3 \). Use the quadratic formula to obtain

\[
z^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{4p^3}{27}}.
\]
Let \( w = z^3 \), so the quadratic equation is

\[
w^2 + qw - \frac{p^3}{27} = 0.
\]

Applying the quadratic formula gives

\[
w = -\frac{q}{2} \pm \sqrt{\frac{q^2 + \frac{4p^3}{27}}{4}}.
\]

Replacing \( w \) by \( z^3 \) gives the desired result.

4. Introduce \( R \) as an abbreviation for \( (\frac{p}{3})^3 + (\frac{q}{2})^2 \) and rewrite the last equality as \( z^3 = -\frac{q}{2} \pm \sqrt{R} \).

Observe that

\[
\sqrt{\frac{q^2 + \frac{4p^3}{27}}{4}} = \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3},
\]

which is just \( \sqrt{R} \). So we have \( z^3 = -\frac{q}{2} \pm \sqrt{R} \) as claimed.

5. There are two possible values for \( z^3 \), namely \(-\frac{q}{2} + \sqrt{R} \) and \(-\frac{q}{2} - \sqrt{R} \). Multiply these two values together and simplify. Show that you get

\[
\left( -\frac{q}{2} + \sqrt{R} \right) \left( -\frac{q}{2} - \sqrt{R} \right) = \left( -\frac{p}{3} \right)^3.
\]

\[
\left( -\frac{q}{2} + \sqrt{R} \right) \left( -\frac{q}{2} - \sqrt{R} \right) = \frac{q^2}{4} - R^2
\]

\[
= -\frac{aq^2}{2} - \frac{q}{2} - \left(\frac{p}{3}\right)^3
\]

\[
= -\left(\frac{p}{3}\right)^3 = \left( -\frac{p}{3} \right)^3.
\]

6. Take cube roots of both sides above and deduce that the two values of \( z \) have a product satisfying

\[
\left( 3\sqrt[3]{-\frac{q}{2} + \sqrt{R}} \right) \left( 3\sqrt[3]{-\frac{q}{2} - \sqrt{R}} \right) = -\frac{p}{3}.
\]

This is obvious, since \( \sqrt[3]{\alpha \beta} = \sqrt[3]{\alpha} \sqrt[3]{\beta} \).

7. Observe that this means that if you choose \( z \) to be the cube root of \(-\frac{q}{2} + \sqrt{R} \), then \(-\frac{p}{3} \) is the cube root of \(-\frac{q}{2} - \sqrt{R} \).

If you choose \( z \) as specified, then the equation in the previous part is just

\[
z \left( 3\sqrt[3]{-\frac{q}{2} - \sqrt{R}} \right) = -\frac{p}{3}.
\]
Then we obviously have

\[
\left(3\sqrt[3]{-\frac{q}{2} + \sqrt{R}}\right) = -\frac{p}{3z}.
\]

8. Recall that \( z \) was introduced to satisfy \( y = z - \frac{p}{3z} \). You have shown that the two terms on the right of this equation, \( z \) and \(-\frac{p}{3z}\), are the cube roots of \( -\frac{q}{2} + \sqrt{R} \) and \(-\frac{q}{2} - \sqrt{R} \), respectively.

9. Conclude that \( y \) is the sum of these two cube roots:

\[
y = 3\sqrt[3]{-\frac{q}{2} + \sqrt{R}} + 3\sqrt[3]{-\frac{q}{2} - \sqrt{R}}.
\]

This is just writing down what \( z \) and \( \frac{p}{3z} \) are after the substitution for \( z \).

• Exercise 10.8: Solve \( y^3 - 3y + 2 = 0 \). Use Cardano’s formula to find one solution \( r \). This should be easy. Then use this solution to factor \( y^3 - 3y + 2 \) and find the other two solutions.

We apply Cardano’s formula with \( p = -3 \) and \( q = 2 \) to get

\[
y = 3\sqrt[3]{\frac{-2}{2} + \sqrt{R}} + 3\sqrt[3]{\frac{-2}{2} - \sqrt{R}}
\]

where

\[
R = \left(\frac{-3}{3}\right)^3 + \left(\frac{2}{2}\right) = (-1)^3 + 1^2 = 0.
\]

So \( y = \sqrt[3]{-1 + 0} + \sqrt[3]{-1 - 0} = -2 \).

This means that \( y^3 - 3y + 2 \) is divisible by \( y + 2 \). Dividing gives

\[
y + 2) y^3 - 3y + 2
\]

\[
= y^3 - 2y^2 + 1
\]

\[
- y^3 - 2y^2
\]

\[
= -2y^2 - 3y
\]

\[
2y^2 + 4y
\]

\[
y + 2
\]

\[
- y - 2
\]

\[
= 0
\]

so the quotient is \( y^2 - 2y + 1 = (y - 1)^2 \). This means that the other root of \( y^3 - 3y + 2 \) is 1, with multiplicity 2.