Math 455—Spring 2020
Solutions for Homework due April 9

Section 2.1:

2 Assume that a vowel is one of the five letters a, e, i, o, or u.

(a) How many eleven-letter sequences from the alphabet contain exactly three vowels?

To construct an eleven-letter sequence with exactly three vowels, first choose the locations which will contain vowels. There are \( \binom{11}{3} \) ways to do this. There are then 5^3 ways to fill those spaces with vowels, and 21^8 ways to fill the other spaces with non-vowels. The total number of sequences is therefore

\[
\binom{11}{3} \cdot 5^3 \cdot 21^8 = 780,096,474,320,625.
\]

(b) How many of these have at least one repeated letter?

The easiest way to compute the number of these sequences with at least one repeated letter is to compute the number which have no repeated letter, and subtract this from the previous answer. This number is

\[
\binom{11}{3} \cdot P(5, 3) \cdot P(21, 8) = \binom{11}{3} \cdot 5^3 \cdot 21^8 = 81,226,696,320,000.
\]

The difference is 698,869,778,000,625.

8. Compute the number of ways to deal each of the following five-card hands in poker.

Note that there are lots of different ways to do structure the calculations. I’m only giving one approach for each part.

(a) Straight: the values of the cards form a sequence of consecutive integers. A jack has value 11, a queen 12, and a king 13. An ace may have a value of 1 or 14, so A 2 3 4 5 and 10 J Q K A are both straights, but K A 2 3 4 is not. Furthermore, the cards in a straight cannot all be of the same suit (a flush).

There are 10 integers that can be the lowest one in a sequence corresponding to a straight. So we can figure out how many ways there are to get a straight starting with some specific integer, say 1 (an ace regarded as low) and multiply by 10

For a straight starting with a low ace, we need one ace, one 2, one 3, one 4, and one 5. There are 4 choices for each
of these (one for each suit), so we have $4^5$ different ways of making the sequence ace, 2, 3, 4, 5. Of these, however, 4 of them have all 5 cards from the same suit, and those don’t count as straights. So there are $4^5 - 4$ ways to get the straight A-5 without all the cards being from the same suit. Then there are $10(4^5 - 4)$ total straights.

(b) **Flush**: All five cards have the same suit (but not in addition a straight).

If we fix one of the 4 suits, any choice of 5 cards from the 13 of that suit will give 5 cards of the same suit. There are $\binom{13}{5}$ ways to do that. But 10 of those are also straights and don’t count. So the answer is

$$4 \left( \binom{13}{5} - 10 \right).$$

(c) **Straight flush**: both a straight and a flush. Make sure that your counts for straights and flushes do not include the straight flushes.

For a given suit, there are 10 straight flushes (because the run of 5 consecutive values can start with anything from 1 to 10). So there are 40 straight flushes.

(e) **Two distinct matching pairs** (but not a full house).

There are $\binom{13}{2}$ ways to choose the two kinds of cards for the pairs, and then $\binom{4}{2} = 6$ ways to choose the two particular cards of each kind. The last card can be any of the 44 remaining cards not of the two kinds making up the pairs. So there are $\binom{13}{2} \cdot 36 \cdot 44$ such hands.

(f) **Exactly one matching pair** (but no three of a kind).

Here we choose 1 of the 13 kinds for the pair, and 3 other kinds for the remaining cards. There are $13 \cdot \binom{12}{3}$ ways to choose the kinds, and for each of these there are $\binom{4}{2}$ ways to choose the 2 cards in the pair and $4^3$ ways to choose one card from each of the other 3 kinds. So there are $13 \cdot \binom{12}{3} \cdot \binom{4}{2} \cdot 4^3$ such hands.

11. Suppose a positive integer $N$ factors as $N = p_1^{n_1} p_2^{n_2} \ldots p_m^{n_m}$, where $p_1, p_2, \ldots, p_m$ are distinct prime numbers and $n_1, n_2, \ldots, n_m$ are all positive integers. How many different positive integers are divisors of $N$?

By the unique factorization theorem for the integers, the positive integers dividing $N$ are exactly the numbers of the form

$$p_1^{a_1} p_2^{a_2} \ldots p_m^{a_m},$$
where $0 \leq a_i \leq n_i$ for $i = 1, \ldots, m$. Furthermore, each of these numbers is different. There are $n_i + 1$ choices for the exponent $a_i$, so the total number of positive divisors is

$$(n_1 + 1)(n_2 + 1) \cdots (n_m + 1).$$

Section 2.2

2. Prove the absorption/extraction identity: If $n$ is a positive integer and $k$ is a nonzero integer, then

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}.$$  

If $k > n$, then $\binom{n}{k} = \binom{n-1}{k-1} = 0$, so the equality holds. If $k < 0$, then $k - 1 < 0$ as well and again both sides are 0. If $k > 0$, we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n((n-1)!)}{k((k-1)!)((n-k)!)!} = \frac{n(n-1)}{k(k-1)}.$$  

4. Suppose that a museum curator with a collection of $n$ paintings by Jackson Pollack needs to select $k$ of them for display, and needs to pick $m$ of these to put in a particularly prominent part of the display. Show how to count the number of possible combinations in two ways so that the cancellation identity appears.

The curator needs to select a set of $k$ paintings from a collection of $n$, and then select a subset of size $m$ from the set of $k$. (So we presumably have $0 < m \leq k \leq n$, on the grounds that curators don’t worry about subsets of size 0.) There are $\binom{n}{k}$ ways to select the set of $k$, and, given a set of $k$, there are $\binom{k}{m}$ ways to select a subset of size $m$. So there are $\binom{n}{k}\binom{k}{m}$ ways to pick the sets of paintings for display.

On the other hand, the curator could first pick the $m$ paintings for prominent display. There are clearly $\binom{n}{m}$ ways to do that. The curator then needs to pick $k - m$ paintings from the remaining $n - m$ Pollacks for the less prominent display. So there are $\binom{n}{m}\binom{n-m}{k-m}$ ways to select the paintings for display.

Since these are two ways of counting the same thing, we get the cancellation identity.

10. In the Virginia lottery game Win For Life, an entry consists of a selection of six different numbers between 1 and 42, and each drawing selects seven different numbers in this range. How many different entries can match at least three of the drawn numbers?
We can think of the drawing as a subset \( S \subseteq \{1, \ldots, 42\} \) with \( |S| = 7 \). An entry that matches exactly 3 of the elements of \( S \) consists of 6 numbers, 3 from \( S \) and 3 from the 35 numbers in the complement of \( S \). So there are \( \binom{7}{3} \binom{35}{3} \) entries that match exactly 3 of the numbers in \( S \). Similar analyses for matching 4, 5, or 6 of the elements of \( S \) then tell us that the number of entries matching at least 3 of the elements of \( S \) is

\[
\binom{7}{3} \binom{35}{3} + \binom{7}{4} \binom{35}{2} + \binom{7}{5} \binom{35}{1} + \binom{7}{6} \binom{35}{0}.
\]

Section 2.3

9e. Compute the number of \( r \)-letter sequences that can be formed from “Walla Walla WA” (ignoring differences in case) for \( r = 4, 5, 12 \).

There are 3 W’s, 5 A’s and 4 L’s, and 12 letters total. We perform an analysis similar to the one at the end of section 2.3, except that since there are only 3 different letters, the pattern \( wxyz \) is not possible.

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<th>Sub-multisets</th>
<th>Orderings</th>
<th>Total</th>
</tr>
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<td>wwwww</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>wwwwx</td>
<td>6</td>
<td>4</td>
<td>24</td>
</tr>
<tr>
<td>wwwxx</td>
<td>3</td>
<td>6</td>
<td>18</td>
</tr>
<tr>
<td>wwxy</td>
<td>3</td>
<td>12</td>
<td>36</td>
</tr>
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</table>

There are 80 total possible four-letter words.

For \( r = 5 \), we get

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<th>Total</th>
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</thead>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>wwwwx</td>
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<td>5</td>
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</tr>
<tr>
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<td>20</td>
<td>60</td>
</tr>
<tr>
<td>wwwwxy</td>
<td>3</td>
<td>30</td>
<td>90</td>
</tr>
</tbody>
</table>

There are 231 possible words.

For 12-letter words, we must use up all the letters available, so this is just asking how many orderings of the letters there are, which is

\[
\binom{12}{3, 5, 4} = 27720.
\]