Section 1.3.1

1. Draw all unlabeled trees of order 7. Hint: There are a prime number of them.

I’m not going to draw them here. There are 11; to find them, you can draw the 6 trees of order 6 and start adding edges.

3. Let $T$ be a tree of order $n \geq 2$. Prove that $T$ is bipartite.

You can do this directly by choosing a vertex $v$ and taking one partite set to be the vertices whose distance from $v$ is even. To show there are no edges between vertices in the same partite set, you can make an argument that if there is such an edge, there has to be a cycle. We can also do this by induction, with less bookkeeping than the cycle argument requires. The result is clearly true for the tree of order 1. So suppose the result is true if $T$ has order less than $k$ and assume $T$ is a tree of order $k$. Let $v$ be a leaf. Then $T - v$ is a tree of order $k - 1$ and therefore bipartite by induction. The unique vertex of $T$ that’s adjacent to $v$ belongs to a partite set of $T - v$. Add $T$ to the other partite set to get the partite sets for $T$.

4. Graphs of the form $K_{1,n}$ are called stars. Prove that if $K_{r,s}$ is a tree, then it must be a star.

Suppose both $r, s > 1$. Choose vertices $a_1 \neq a_2 \in A$ and $b_1 \neq b_2 \in B$. Then the path $a_1, b_1, a_2, b_2, a_1$ is a cycle. But a tree has no cycles. It follows that at least one of $r$ and $s$ must be 1.

Section 1.3.2

1. Draw each of the following if you can. If you cannot, explain the reason.

(a) A 10-vertex forest with exactly 12 edges.

This is impossible; a forest can’t have more edges than vertices.

(b) A 12-vertex forest with exactly 10 edges.

A forest of order $n$ with $k$ components has $n - k$ edges.

So take one component to be a tree of order $a$ and the second component to be a tree of order $b$ where $a + b = 12$.

(c) A 14 vertex forest with exactly 14 edges.
This is impossible; a forest must have fewer edges than vertices.

(d) A 14-vertex forest with exactly 13 edges.

(e) A 14-vertex forest with exactly 12 edges.

As above, we want a forest with $n$ vertices and $n - 2$ edges. So we need two trees whose orders sum to 14.

2. Suppose a tree $T$ has an even number of edges. Show that at least one vertex must have even degree.

If $T$ has $m = 2k$ edges, it has $2k + 1$ vertices. We know that the number of edges of odd degree in a graph must be even (since the sum of the degrees is twice the size). But $T$ has an odd number of vertices, so they can't all have odd degree.

3. Let $T$ be a tree with max degree $\Delta$. Prove that $T$ has at least $\Delta$ leaves.

Suppose $T$ has order $n$. Then it has $n - 1$ edges and the sum of the degrees of $T$ is $2(n - 1)$. Let $v$ be a vertex of degree $\Delta$. If $T$ has $m$ leaves, with $m < \Delta$, there are at $n - m - 1$ vertices other than $v$ that have degree at least 2. Then the sum of the degrees is at least $\Delta + 2(n - m - 1) + m = \Delta + 2n - m - 2$. Since $m < \Delta$, this is strictly greater than $2(n - 1)$.

5. Prove that a graph is a tree if and only if for every pair of vertices $u, v$, there is exactly one path from $u$ to $v$.

If there is exactly one path from $u$ to $v$, for every $u, v \in V(G)$, then $G$ is connected (there is a path) and acyclic (since if there were a cycle, there would be vertices with two paths between them). So one direction is easy.

The other direction is harder. Here are two ways of doing that:

First, we can argue by induction on the order $n$ of $T$. If $n = 1$ or 2, it's easy to see that there's only one path between any two vertices. So assume that the result holds for all trees of order $k \geq 2$ and let $T$ be a tree of order $k + 1$. We know that $T$ has at least two leaves, so let $v$ be a leaf of $T$ and let $w$ be the (unique) vertex in $T$ that is adjacent to $v$. We know that $T - v$ is a tree of order $k$. If $x$ and $y$ are vertices of $T - v$, then any path in $T$ from $x$ to $y$ must not contain $v$ because it would have to contain $w$ twice. So all the paths from $x$ to $y$ in $T$ are really paths in $T - v$. By the induction hypothesis, there is exactly one such path. How about paths from $x$ to $v$? Any such path must contain $w$ and end with the edge $wv$. So it must be composed of a path from $x$ to $w$, followed by the vertex $v$. 
Since there is exactly one path from $x$ to $w$, this shows there is exactly one path from $x$ to $v$. Since this is true for any vertex $x$ of $T - v$, we’ve shown that there is exactly one path between any two distinct vertices of $T$.

A second argument starts by supposing that we have distinct vertices $u, v$ with two paths from $u$ to $v$, and deriving a contradiction. Let $P : u = v_1, \ldots, v_m = v$ be one path and $Q : u = w_1, \ldots, w_n = v$ be the other. We will show that there must be a cycle in $T$, contradicting the assumption that $T$ is a tree.

If the only vertices the two paths have in common are $u$ and $v$, then we can easily make a cycle: $u, v_2, \ldots, v_{m-1}, v, w_n, \ldots, w_2, u$. So assume that the paths have some other vertex in common. Let $i$ be the smallest index greater than 1 such that the edge $v_{i-1}v_i$ of the path $P$ is not in the path $Q$. So $v_k = w_k$ for $k < i$.

Now let $j$ be the smallest index greater than $i - 1$ such that $v_j$ is in $Q$. So $v_j = w_\ell$ for some $\ell$. Further, $\ell > i$, since the edge $v_{i-1}v_i$ isn’t in $Q$.

The two paths $v_{i-1}, v_i, \ldots, v_j$ and $w_{\ell-1}, \ldots, w_\ell$ have no vertices in common other than $v_{i-1} = w_{\ell-1}$ and $v_j = w_\ell$; if they did, then $j$ wouldn’t be the smallest index greater than $i$ – such that $v_j$ is in $Q$. So we have a closed path $v_{i-1}, v_i, \ldots, v_j = w_\ell, w_{\ell-1}, \ldots, w_1$. The length of this path is at least 3 (since $\ell > i$), so it is a cycle.

6. Prove that $T$ is a tree if and only if $T$ contains no cycles, and for any new edge $e$, the graph $T + e$ has exactly one cycle.

Suppose $T$ is a tree. Then $T$ contains no cycles, so we only need to show that for any new edge, $T + e$ has exactly one cycle. Adding an edge to any connected graph creates a cycle (make sure you understand why); we have to show that $T + e$ has only one cycle. Suppose $T + e$ has two cycles. They must both contain $e$, or one of them would have been a cycle in $T$. Let $e = uv$, so $u$ and $v$ are on two cycles and there must be two paths from $u$ to $v$. By walking along one of the in the $v$ to $u$ direction and the other in the $u$ to $v$ direction, we get a closed walk. As we’ve done in class, we can find a cycle contained in that.

For the converse, suppose $T$ is a graph that contains no cycles and, for any new edge $e$, $T + e$ contains exactly one cycle. Since $T$ contains no cycles, we know that if $n > 2$, $T$ is not $K_n$ so there’s at least one edge that could be added. (If $T$ is $K_1$ or $K_2$, we know $T$ is a tree.) A connected, acyclic graph is a tree, so we need to show that $T$ is connected. But if $T$ is not connected, choose vertices $v$ and $w$ in different connected components of $T$.
and add the edge $vw$. But then $T + e$ doesn’t contain a cycle: any cycle must go through the new edge $uv$ (or it would have been a cycle in $T$), but any path that goes through $uv$ must end in a different connected component than it started it from and so can’t be a cycle. This contradicts the hypothesis that adding an edge adds exactly one cycle, so $T$ must be connected.

Extra credit problem for 1.3.2:

7. Show that every nonleaf in a tree is a cut vertex.

A nonleaf is a vertex of degree greater than 1. Suppose that $v$ is a nonleaf vertex of a tree $T$. We need to show that $T - v$ is disconnected. Since the degree of $v$ is at least 2, let $u$ and $w$ be vertices that are adjacent to $v$ in $T$. Then $u, v, w$ is a path in $T$ from $u$ to $w$. If there were a path $P$ from $u$ to $w$ in $T - v$, it would not include $v$, so $P, v, u$ would be a cycle in $T$. (Or, using problem 5, we would have 2 paths from $u$ to $w$ in $T$, which is impossible.)