Section 1.1.2

13. Prove or disprove: If every vertex of a connected graph $G$ lies on at least one cycle, then $G$ is 2-connected.

This is false. Consider a “figure eight” graph formed by connecting two copies of $K_3$ at a shared vertex.

14. Prove that every 2-connected graph contains at least one cycle.

Suppose $G$ is a 2-connected graph (and therefore a connected graph). It follows that there is no cut vertex in $G$—at least two vertices have to be removed to disconnect $G$—and that $G$ has at least 3 vertices. Let $x \in V(G)$. Since $G$ is 2-connected, we must have $\deg(x) \geq 2$. (Why?) So there are distinct vertices $y$ and $z$ adjacent to $x$. Since $G$ is 2-connected, $G - x$ is connected, and there is a path $y = v_1, v_2, \ldots , v_k = z$ in $G - x$. Then $x, y, v_2, \ldots , v_{k-1}, z, x$ is a cycle in $G$.

Another way to do this is to show that there’s no bridge, and apply problem 11 from Section 1.1.2 that says that an edge is a bridge if and only if it lies on no cycle. So if there are no bridges, each edge must lie on a cycle. To see that there are no bridges, suppose to the contrary that some edge $uv$ is a bridge. Then $G - e$ has more connected components than $G$; since $G$ is connected, this means that $G - e$ is disconnected. So there are some vertices that are connected in $G$ only by walks that contain the edge $e$. But $G - e$ disconnected implies that $G - u$ is disconnected as well since the edge $e$ is removed when we remove $u$. Then $\{u\}$ would be a vertex cut set of size 1, contradicting the hypothesis that $G$ is 2-connected.

Section 1.2.1

1. Find the radius, diameter, and center of the graph shown in Figure 1.28.

The maximal eccentricity is 6, attained at the lower left and lower right vertices. The minimal eccentricity is 3, at the middle vertex in the middle row. So the radius is 3, the diameter is 6, and the center is the single vertex in the middle of the middle row.

4. If $x$ is in the periphery of $G$ and $d(x, y) = \text{ecc}(x)$, then prove that $y$ is in the periphery of $G$. 
Since $x$ is in the periphery of $G$, we know that $\text{ecc}(x)$ is equal to the maximum value of the eccentricity of any vertex. Of course, $\text{ecc}(y)$ must be at least $d(x, y)$. But $d(x, y) = \text{ecc}(x)$ and $x$ in the periphery means that $\text{ecc}(y)$ can’t be greater than $\text{ecc}(x)$. So $\text{ecc}(y) = \text{ecc}(x)$ and $y$ is in the periphery of $G$.

5. If $u$ and $v$ are adjacent vertices in a graph, prove that their eccentricities differ by at most one.

Let $x$ be an arbitrary vertex of the graph and let $v = v_1, v_2, \ldots, v_k = x$ be a shortest path from $v$ to $x$, so that $d(v, x) = k - 1$. Then $u, v, v_2, \ldots, v_k = x$ is a walk from $u$ to $x$ of length $k = d(v, x) + 1$. Since every walk contains a path, there must be a path from $u$ to $x$ of length at most $d(v, x) + 1$, so $d(u, x) \leq d(v, x) + 1$. Taking a shortest path from $u$ to $x$, the same argument with the roles of $u$ and $v$ reversed shows that $d(v, x) \leq d(u, x) + 1$. So we have $-1 \leq d(u, x) - d(v, x) \leq 1$, or $|d(u, x) - d(v, x)| \leq 1$.

Since this is true for every vertex $x$ in the graph and the eccentricity of a vertex is the maximum distance from that vertex to any vertex in the graph, it must be the case that the eccentricities of $u$ and $v$ differ by at most 1.

Extra credit problem for 1.2.1:

10 Let $G$ be a connected graph with at least one cycle. Prove that $G$ has at least one cycle whose length is less than or equal to $2\text{diam}(G) + 1$.

Let $d$ be the diameter of $G$ (since $G$ is connected we know there is a diameter) and suppose that all cycles in $G$ have length greater than $2d + 1$. Let $v_1, \ldots, v_r, v_1$ be a cycle of shortest length $r$.

Consider the paths $P_1 = v_1, \ldots, v_{d+2}$ and $P_2 = v_{d+2}, v_{d+3}, \ldots, v_r, v_1$. Each of these has length at least $d + 1$. But there is a path $P'$ of length at most $d$ from $v_{d+2}$ to $v_1$. If the only vertices on $P'$ that are also on $P_1$ are $v_1$ and $v_{d+2}$, then $P_1$ followed by $P'$ is a cycle of length $2d + 1$. If some other vertex is on both $P_1$ and $P$, we can find a shorter cycle.

Section 1.2.2

3. If $A$ is the adjacency matrix for the graph $G$, show that the $(j, j)$ entry of $A^2$ is the degree of $v_j$.

We have seen that the $(j, j)$ entry of $A^2$ is the number of walks of length 2 from $v_j$ to $v_j$. But a walk of length two from $v_j$ to $v_j$ is a sequence $v_j, v_k, v_j$, where $v_j v_k$ is an edge. So the number of such walks is exactly the number of vertices $v_k$ such that $v_j v_k$ is an edge. This is the degree of $v_j$.  

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4. Let $A$ be the adjacency matrix for the graph $G$.

(a) Show that the number of triangles that contain $v_j$ is $\frac{1}{2}[A^3]_{j,j}$.

A triangle in $G$ is a cycle consisting of 3 vertices. Any walk of length 3 from $v_j$ to $v_j$ must be a cycle, since if it only involved 2 vertices it couldn’t return to $v_j$ in an odd number of steps. So we are asked to count the number of distinct cycles of length 3 that include $v_j$. We know that the $(j,j)$ entry of $A^3$ counts the number of walks of length 3 from $v_j$ to $v_j$. Each triangle containing $v_j$ gives us 2 such walks, since we can go around the triangle in two different orders. So the number of triangles containing $v_j$ is $\frac{1}{2}[A^3]_{j,j}$.

(b) The trace of a square matrix $M$, denoted $\text{Tr}(M)$, is the sum of the entries on the main diagonal. Prove that the number of triangles in $G$ is $\frac{1}{6} \text{Tr}(A^3)$.

By part (a), $\frac{1}{2} \text{Tr}(A^3)$ is the sum over all vertices $v_j$ of the number of triangles containing $v_j$. Since each triangle contains 3 vertices, this sum counts each triangle 3 times. So $\frac{1}{6} \text{Tr}(A^3)$ is the number of triangles in $G$. 