Section 1.1.2

3. Consider the graph shown in Figure 1.10. [This is $K_5$.]

(a) How many different paths have c as an end vertex?

Note that, according to the definition of end vertex on page 6, the end vertices of a walk $v_1, \ldots, v_k$ are $v_1$ and $v_k$, so an end vertex can be at either end of the walk.

Let’s start by counting the number of paths starting at c.

There is only one path of length 0 starting at c, the vertex c by itself. There are 4 paths starting at c of length 1, c, x, where x is one of a, b, d, e. Given any path c, x, we can extend it to c, x, y in 3 ways; y must be a vertex other than c and x. That means there are a total of 12 paths of length 2 starting at c. And given a path c, x, y, we can extend it to c, x, y, z in 2 ways, since there are two vertices not equal to c, x, or y. Extending the 12 paths of length 2, we therefore get 24 paths of length 3. And each of those paths can be extended in exactly one way to a path c, x, y, z, w of length 4, so there are 24 paths starting at c of length 4. Thus there are 1 + 4 + 12 + 24 + 24 = 65 paths starting at c.

Now, any path starting at c and ending at a vertex $x \neq c$ also determines a path starting at $x$ and ending at c, by reversing the order. So that gives two paths having c as an end vertex. So for the total number of paths having c as an end vertex, we have $1 + 2(4 + 12 + 24 + 24) = 129$.

(b) How many different paths avoid vertex c altogether?

Since every vertex is adjacent to c, any path avoiding c altogether can be made into a path starting at c, just by sticking c on at the beginning. That means that the number of paths avoiding c altogether is the same as the number of paths of length at least 1 that start at c. We have seen that this number is 64.

(c) What is the maximum length of a circuit in this graph? Give an example of such a circuit.

The maximum length is 10. Here’s one: $a, b, c, d, e, a, c, e, b, d, a$

(d) What is the maximum length of a circuit that does not include vertex c? Give an example of such a circuit.

This is the same as the maximum length of a circuit in $K_4$, which is 4. An example is $a, b, d, e, a$. 


4. Is it true that a finite graph having exactly two vertices of odd degree must contain a path from one to the other? Give a proof or a counterexample.

   It is true. Suppose \( v \) and \( w \) are the two vertices of odd degree in a graph \( G \) with exactly two such vertices. If there is not a path from one to the other, then they must lie in different connected components. Let \( G_1 \) be the connected component containing \( v \). By Theorem 1.1, \( G_1 \) must have an even number of vertices of odd degree, so it has a vertex \( u \) of odd degree with \( u \neq v \). But \( u \) can’t equal \( w \), since \( w \) is not a vertex of \( G_1 \). So there are more than 2 vertices of odd degree, contradicting our hypothesis. This says there must be a path between \( v \) and \( w \).

5. Let \( G \) be a graph where \( \delta(G) \geq k \).

   (a) Prove that \( G \) has a path of length at least \( k \).

   Let \( P = v_1, v_2, \ldots, v_{m+1} \) be a path in \( G \) with length \( m < k \). The degree of \( v_{m+1} \) is at least \( k \), so there are \( k > m \) vertices adjacent to \( v_{m+1} \). At least one of those vertices does not occur on \( P \), since only \( m \) there are only \( m \) vertices other than \( v_{m+1} \) on \( P \). Take \( v_{m+2} \) to be a vertex adjacent to \( v_{m+1} \) with \( v_{m+1} \neq v_i \) for \( i = 1, \ldots, m \). Then \( P' = v_1, v_2, \ldots, v_{m+1}, v_{m+2} \) is a longer path. So as long as we have a path of length less than \( k \), we can extend that path to a longer one. This shows that there must be a path of length at least \( k \).

   (b) If \( k \geq 2 \) prove that \( G \) has a cycle of length at least \( k + 1 \).

   Let \( P = v_1, \ldots, v_{m+1} \) be a path of maximal length in \( G \). Every vertex adjacent to \( v_1 \) has to be on \( P \), since otherwise we could extend \( P \) to a path \( v_0, v_1, \ldots, v_{m+1} \). But there are at least \( k \) vertices adjacent to \( v_1 \), so one of those vertices must be \( v_i \) for some \( i \geq k + 1 \). Then \( v_1, \ldots, v_i, v_1 \) gives a cycle of length at least \( k + 1 \).

8. Let \( P_1 \) and \( P_2 \) be two paths of maximum length in a connected graph \( G \). Prove that \( P_1 \) and \( P_2 \) have a common vertex.

   Let \( P_1 = v_1, \ldots, v_m \) and \( P_2 = w_1, \ldots, w_m \) be two paths of maximal length in \( G \) and assume they have no vertex in common, so \( v_i \neq w_j \) for all \( i, j \). Since \( G \) is connected, there exists a path \( Q = v_1 = u_1, u_2, \ldots, u_r = w_m \) from \( v_1 \) to \( w_m \). Let \( i \) be the maximum index such that \( u_i \) is on \( P_1 \), and let \( j \) be the minimum index such that \( j > i \) and \( u_j \) is on \( P_2 \). Note that \( i \) can’t be 1 or \( m \), since then we would have a longer path \( u_2, v_1, v_2, \ldots, v_m \) or \( v_1, v_2, \ldots, v_m, u_m+1 \), and similarly for \( j \). Let \( Q^* = u_{i+1}, u_{i+2}, \ldots, u_{j-1} \). (If \( i + 1 = j \), \( Q^* \) is empty.)
9. Let $G$ be a graph of order $n$ that is not connected. What is the maximum size of $G$?

The answer is $\frac{(n-1)(n-2)}{2}$. There are several ways to prove this.

One way is to argue that the disconnected graph of order $n$ with the most edges must have each connected component being a complete graph. The size of $K_m$ is $\binom{m}{2} = \frac{m(m-1)}{2}$. So if the orders of the connected components are $m_1, \ldots, m_k$ for some $k$, we’re interested in finding the positive integers $m_1, \ldots, m_k$ with $m_1 + \ldots + m_k = n$ that maximize $\sum_i \frac{m_i(m_i-1)}{2}$. Various optimization techniques will show that $m_1 = n-1$ and $m_2 = 1$ gives the maximum, which is $\frac{(n-1)(n-2)}{2}$.

We can also prove this by induction. We’ll take the base case to be $n = 2$ (since every graph of order 1 is connected). It’s clear that the maximum size of a disconnected graph of order 2 is 0 = $\frac{(2-1)(2-2)}{2}$. So suppose that the maximum size of a disconnected graph of order $k$ is $\frac{(k-1)(k-2)}{2}$ and consider a graph of order $k+1$. We want to show that any graph of order $k+1$ and size strictly greater than $\frac{(k-1)(k-2)}{2}$ is connected.

So suppose $G$ is a graph of order $k+1$ and size $s$ greater than $\frac{k(k-1)}{2}$. We will show that $G$ is connected. Let $v$ be a vertex of minimal degree in $G$. The degree of $v$ must be at least 1, or $G - v$ would be a graph of order $k$ and more than $\frac{k(k-1)}{2}$ edges, which is impossible. If every vertex of $G$ has degree $k$, $G = K_{k+1}$ and we know $G$ is connected. So we can assume that $\deg(v) < k$. Then $G - v$ has at least $s - (k-1)$ edges and $k$ vertices. But

$$s - (k-1) > \frac{k(k-1)}{2} - (k-1) = \frac{k^2 - 3k + 2}{2} = \frac{(k-1)(k-2)}{2}$$

so $G - v$ is a graph of order $k$ with at least $\frac{(k-1)(k-2)}{2}$ edges. By induction, $G - v$ is connected.

3
Since \( \deg(v) > 1 \), there must be an edge in \( G \) from \( v \) to some vertex of \( G - v \), and that vertex is connected to every other vertex of \( G - v \). So \( G \) is connected. This completes the induction proof.

10. Let \( G \) be a graph of order \( n \) and size strictly less than \( n - 1 \). Prove that \( G \) is not connected.

Suppose that \( G \) is connected, with \( n \) vertices. Start with some vertex \( v_1 \). Since \( G \) is connected, there must be a vertex \( v_2 \) with an edge \( v_1v_2 \). Then (if \( n > 2 \)), there must be a vertex \( v_3 \neq v_1, v_2 \) that is adjacent to either \( v_1 \) or \( v_2 \). Since \( G \) is connected, so there is a walk from \( v_1 \) to each vertex in the graph, we can keep going this way, finding another vertex that is adjacent to one of the ones we’ve already listed, until we’ve used up all the vertices. But each time we add a vertex, we must add at least one new edge. Since we have at most \( n - 2 \) edges, we can’t get all \( n \) vertices. This is a contradiction, so \( G \) can’t be connected.

You can also do this by induction. The smallest \( n \) such that the size can be strictly less than \( n - 1 \) is 2, so for the base case, assume \( n = 2 \). Then the size is 0, so the graph has 2 vertices and 0 edges. That’s clearly disconnected. So assume the the result is true if the order of \( G \) is at least 2 and less than \( k \). Consider a graph \( G \) of order \( k \) with size strictly less than \( k - 1 \). The sum of the degrees of the vertices of \( G \) is therefore less than \( 2(k - 1) \), so at least one vertex \( v \) has degree less than 2. If \( v \) has degree 0, it’s not adjacent to any other vertex and \( G \) must be disconnected. If \( v \) has degree 1, it’s adjacent to exactly one other vertex \( u \). So \( G - v \) has \( k - 1 \) vertices and (because the edge \( vu \) is removed when we delete \( v \)) strictly less than \( k - 2 \) edges. By the induction hypothesis, \( G - v \) is disconnected. But if \( x \) and \( y \) are in different connected components of \( G - v \), they must also be in different connected components of \( G \): adding back the vertex \( v \) and the single edge \( vu \) can’t create a path from \( x \) to \( y \).