1. Give precise and complete mathematical definitions for the following:
   (a) (5 points) An edge $e$ in a graph $G$ is a bridge if . . .

   **Solution:** the graph $G - e$ has more components than $G$.

   (b) (5 points) A spanning tree for a graph $G$ is . . .

   **Solution:** a subgraph $T$ of $G$ such that $T$ is a tree and contains every vertex of $G$.

   (c) (5 points) A Hamiltonian path in a graph $G$ is . . .

   **Solution:** a path in $G$ that contains all the vertices of $G$.

   (d) (5 points) A graph is $k$-connected if . . .

   **Solution:** $\kappa(G) \geq k$.

2. For the graph shown below (and the indicated numbering of the vertices)

   ![Graph Diagram]

   (a) (7 points) Give the adjacency matrix.

   **Solution:**

   \[
   \begin{bmatrix}
   0 & 1 & 1 & 1 & 0 \\
   1 & 0 & 0 & 0 & 1 \\
   1 & 0 & 0 & 0 & 1 \\
   1 & 0 & 0 & 0 & 1 \\
   0 & 1 & 1 & 1 & 0 \\
   \end{bmatrix}
   \]

   (b) (4 points) What is the diameter of this graph?

   **Solution:** Each vertex has eccentricity 2, so the diameter is 2.

   (c) (4 points) What is the periphery of this graph?

   **Solution:** All the vertices have the same eccentricity, so the whole graph is the periphery.
3. Make sure to explain your answers on these.

(a) (5 points) Give an example to show that not every trail is a path.

**Solution:** In the graph

![Graph](image)

The walk $v_1, v_2, v_3, v_4, v_2, v_5$ is a trail that is not a path, since the vertex $v_2$ is repeated, but no edge is repeated.

(b) (5 points) Give an example of a graph that is traceable but not Hamiltonian.

**Solution:** The path $P_3$:

![Path P3](image)

has a Hamiltonian path $v_1v_2v_3$ and is therefore traceable, but it has no cycles at all, so it certainly has not Hamiltonian cycle.

4. (15 points) Use Kruskal’s algorithm to find a minimum weight spanning tree for the weighted graph shown below. (Make sure you show enough work for me to tell how you were selecting edges, in what order, etc. And carefully identify the tree at the end.)

![Weighted Graph](image)

**Solution:**

1. There’s a unique edge of minimum weight, $v_2v_4$, so mark it.

2. The marked edges do not form a spanning tree. The minimum weight among unmarked edges is 2, and neither of the edges of weight 2 forms a cycle with the marked edges. So we mark one of them, say $v_2v_3$. (You could mark the other one.)
3. The marked edges do not yet form a spanning tree. The minimum weight among unmarked edges is 2 and the unmarked edge of weight 2 does not form a cycle with the marked edges. So we mark that edge, $v_4v_5$.

4. The marked edges do not yet form a spanning tree. The minimum weight among unmarked edges is 3, but each of the edges $v_2v_5$ and $v_3v_4$ would form a cycle with the marked edges. We can mark any of the other edges of weight 3, all of which are incident with $v_1$. I’ll pick $v_1v_5$.

5. The marked edges form a spanning tree, so we’re done. The tree is

\begin{center}
\begin{tikzpicture}

\node (v1) at (0,0) {$v_1$};
\node (v5) at (-1,-1) {$v_5$};
\node (v2) at (1,-1) {$v_2$};
\node (v4) at (-1,-2) {$v_4$};
\node (v3) at (1,-2) {$v_3$};

\draw (v1) -- (v5);
\draw (v1) -- (v2);
\draw (v2) -- (v3);
\draw (v2) -- (v4);
\draw (v4) -- (v5);
\end{tikzpicture}
\end{center}

and has weight 8.

5. (20 points) Show that a graph $G$ is acyclic if and only if every induced subgraph of $G$ contains a vertex of degree at most 1.

**Solution:** Suppose that $G$ is acyclic. Let $H$ be an induced subgraph of $G$, so $H$ consists of some subset $V(H)$ of the vertices of $G$ and all the edges of $G$ with both end vertices in that set. It’s clear that $H$ is acyclic, since any cycle in $H$ would have to be a cycle in $G$. Therefore each connected component of $H$ is a tree and $H$ is a forest.

We can restrict our attention to a single connected component, which is a tree. A tree of order at least 2 has at least 2 leaves (vertices of degree 1). If the component has order 1, then it doesn’t have any edges so it has a vertex of degree 0. So every induced subgraph $H$ has at least one vertex of degree less than or equal to 1.

For the converse, assume that every induced subgraph of $G$ contains a vertex of degree at most 1 but $G$ has a cycle $C = v_1v_2\ldots v_kv_1$. Let $G'$ be the induced subgraph whose vertex set is $\{v_1, \ldots, v_k\}$. The degrees of the vertices in $G'$ are all at least 2 (there might be additional edges in the induced subgraph, but the degrees can’t be less than the two needed to make a cycle). This contradicts the assumption that every induced subgraph of $G$ has a vertex of degree at most 1, so $G$ must be acyclic.
6. (20 points) Recall that the complement $\overline{G}$ of a graph $G$ is the graph with the same vertex set as $G$ and edge set that is the complement of the edge set of $G$ (so $\{u, v\}$ is an edge of $\overline{G}$ if and only if it is not an edge of $G$). Let $G$ be a connected regular graph that is not Eulerian and suppose that $\overline{G}$ is connected. Show that $\overline{G}$ is Eulerian.

**Solution:** Suppose that $G$ is $r$-regular and has order $n$, so the sum of the degrees of the vertices of $G$ is $nr$. The degree $r$ must be odd since $G$ is not Eulerian, but the sum of the degrees is twice the number of edges and so must be even. Therefore $n$ is even.

Each vertex of $\overline{G}$ has degree $n-1-r$. Since $n$ is even and $r$ is odd, this is even. Then $\overline{G}$ is Eulerian.

7. (20 points (Extra Credit)) **[Since this problem is for extra credit, I will give very little partial credit. So don’t attempt this problem until you’ve done everything you can on the other problems.]** Suppose $G$ is a graph of order at least 3. Show that $G$ is connected if and only if there exist distinct vertices $u, v$ of $G$ such that both $G - u$ and $G - v$ are connected. (Hint: If $G$ is connected, take $u$ and $v$ as far apart as possible and assume that $G - u$ isn’t connected. Show that this leads to a contradiction.)

**Solution:** Suppose $G$ is connected and choose $u, v$ such that $d(u, v) = \text{diam}(G)$. We show that both $G - u$ and $G - v$ are connected. If not, at least one is disconnected; we may suppose that $G - u$ is disconnected. Then $G - u$ contains vertices $x$ and $y$ with no walk in $G - u$ from $x$ to $y$. But $G$ is connected, so there is a path in $G$ from $v$ to $x$ and a path in $G$ from $v$ to $y$. Let $P$ be a shortest path in $G$ from $x$ to $v$, and let $P'$ be a shortest path in $G$ from $v$ to $y$. Since $d(u, v)$ is the diameter of $G$, $u$ cannot lie on $P$ or $P'$ (note that $u \neq x, y$). So $P$ followed by $P'$ gives a walk in $G - u$ from $x$ to $y$. Then there is an $x - y$ walk in $G - u$, contradicting our choice of $x$ and $y$. It follows that both $G - u$ and $G - v$ must be connected. (Another nice way to prove this direction is the following: Let $T$ be a spanning tree for $G$. Since $G$ has order at least 3, we know that $T$ has at least two leaves. Let $u, v$ be leaves of $T$. Then $T - v$ is a tree, and hence a spanning tree for $G - v$, so $G - v$ is connected. Similarly, $T - u$ is a tree, so $G - u$ is connected.)

For the converse, let $u$ and $v$ be distinct vertices with both $G - u$ and $G - v$ connected. Let $x, y$ be two vertices of $G$; we need to show there is an $x - y$ walk in $G$. If $\{x, y\} \neq \{u, v\}$, we may (renaming if necessary) assume that $u \notin \{x, y\}$. So $x$ and $y$ are vertices in $G - u$ and there is an $x - y$ walk in $G - u$. But then there is certainly an $x - y$ walk in $G$. So it only remains to show that there is a $u - v$ walk in $G$. Since the order of $G$ is at least 3, there is a vertex $w$ in $G$ with $w \neq u, v$. Since $G - v$ is connected, there is a $u - w$ walk in $G - v$, and thus in $G$. And since $G - u$ is connected, there is a $w - v$ walk in $G - u$, and thus in $G$. It follows that there is a $u - v$ walk in $G$. 
