Updates

- Exam will be 2.5 hours in window from 8:00 am EDT Tuesday, April 21 to 8:00 am EDT Wednesday, April 22. It will be on Gradescope.

- I will post a review sheet as I did for the first exam and we will review in class on Thursday. If you can’t participate in the Zoom session Thursday to ask questions, send your questions to me and/or Cristian by email. If needed, we can set up separate Zoom sessions to meet with you.

- First two project presentations will be in class-time on Tuesday, April 28. Send questions, difficulties, etc., about projects to me and/or Cristian as well.

- Short homework assignment due Thursday.
Using generating functions

We’ve looked at a couple of examples where the use of generating functions lets us get formulas for the answers to combinatorial questions in ways that are easier and more general than brute force enumeration (though there are often some clever computations that need to be done even with the generating functions).

I want to present one more problem that we can solve with generating functions. This comes from an article by George Pólya in the *American Mathematical Monthly* in 1956. The particular problem is a little contrived, but I think you’ll be able to see how it’s an example of a large class of problems with many applications.

get generating functions recursively
Pólya’s problem is as follows:

*Given pennies, nickels, dimes, quarters, half-dollars, and dollar coins, how many ways can we make a payment of $1? (More generally, given objects of a number of different weights or lengths, how many ways can we build up a particular weight or length?)*

Let $a_k$ be the number of ways of making $k$ cents, so the number of ways of paying $1$ is $a_{100}$. And let the generating function for the sequence $\{a_k\}_k$ be $A(x) = \sum_{k} a_k x^k$.

We want to derive an expression for $A(x)$ by considering the ways we can use different coins.

*use $A(x)$ to determine $a_k$*
If we can only use pennies ...

Suppose we only have pennies. Then there’s only one way to make \( k \) pennies. So if we let \( p_k \) be the number of ways to make \( k \) pennies, we have \( p_k = 1 \) for all \( k \geq 0 \). So if \( P(x) \) is the generating function for \( \{p_k\}_k \), we have

\[
P(x) = x^0 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.
\]
If we can only use nickels . . .

Suppose we only have nickels. Obviously, there aren’t any ways to make amounts that aren’t multiples of 5. If \( n_k \) is the number of ways to make \( k \)¢ with nickels, we have

\[
n_k = \begin{cases} 
1 & \text{if } k \equiv 0 \pmod{5}, \\
0 & \text{otherwise.}
\end{cases}
\]

So the generating function for making \( k \)¢ with nickels alone is

\[
x^0 + x^5 + x^{10} + x^{15} + \ldots = \frac{1}{1 - x^5}. 
\]

Let \( z = x^5 \). LHS is \( \frac{1}{1-z} \).

\[
\begin{align*}
S &= \sum_{i=0}^{\infty} z^i \\
&= 1 + z + z^2 + \ldots \quad \text{LHS} \\
2S &= 2 + z + z^2 + \ldots \\
S - 2S &= 1 \\
S &= \frac{1}{1-z}
\end{align*}
\]
If we can use pennies and nickels...

Suppose we can use pennies and nickels. So, for instance, we can make 6¢ two ways, with 1 nickel and 5 pennies, or with 6 pennies. (Note that we don’t distinguish the order in which you choose the coins.) More generally, if the generating function for the number of ways to make \( k \)¢ with combinations of pennies and nickels is \( C_{p,n}(x) \), this function is

\[
C_{p,n}(x) = P(x)(1 + x^5 + x^{10} + \ldots) = \frac{1}{(1-x)(1-x^5)}.
\]

In general in this way we’d get

\[
A(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})(1-x^{100})}.
\]

But finding the coefficient of \( x^k \) in the Maclaurin series for this is hard.
The number of ways to make 12¢ with pennies and nickels is coefficient of \( x^{12} \) in \((1 + x + x^2 + \ldots)(1 + x^5 + x^{10} + \ldots)\).

The number of ways to make \( k \)¢ with pennies and nickels is coefficient of \( x^k \) in that product.

\[
\frac{1}{1-x} \quad \frac{1}{1-x^5}
\]

Coefficient of \( x^k \) in \( \frac{1}{1-x} \cdot \frac{1}{1-x^5} \) generating function for the number of ways to make \( k \)¢ using pennies and nickels.
From the last slide, we have

\[ C_{p,n}(x) = \frac{P(x)}{1-x^5} = \left( \frac{1}{1-x} \right) \left( \frac{1}{1-x^5} \right) \]

so

\[ C_{p,n}(x)(1-x^5) = P(x) \quad \text{and} \quad C_{p,n}(x) = P(x) + x^5 C_{p,n}(x) \]

Equating coefficients here and writing \( n_k \) for the coefficient of \( x^k \) in \( C_{p,n}(x) \), we get

\[ n_k = \begin{cases} p_k & \text{if } 0 \leq k \leq 4, \\ p_k + n_{k-5} & \text{if } k \geq 5. \end{cases} \]

Coeff of \( x^n \) in \( C_{p,n}(x) \) = coeff of \( x^n \) in \( P(x) \)

+ coeff of \( x^{n+5} \) in \( C_{p,n}(x) \)

Earlier coeff.
Let $C_{p,n,d}(x)$ be the generating function for the number of ways to make $kC$ with pennies, nickels, and dimes. The generating function for using just dimes is $\frac{1}{1-x^{10}}$, so the generating function for using pennies, nickels, and dimes is

$$C_{p,n,d}(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})} = \frac{C_{p,n}(x)}{1-x^{10}}.$$

Multiplying both sides by $1-x^{10}$ as before, we get

$$C_{p,n,d}(x) = C_{p,n}(x) + x^{10}C_{p,n,d}(x).$$
Then, if we let $d_k$ be the coefficient of $x^k$ in $C_{p,n,d}(x)$, we get

$$d_k = \begin{cases} 
\frac{n_k}{n_k + d_{k-10}} & \text{if } 0 \leq k \leq 9, \\
\frac{n_k}{n_k + d_{k-10}} & \text{if } k \geq 10.
\end{cases}$$

The idea is that, if $k < 10$, we must use only pennies and nickels. If $k \geq 10$, we can use no dimes, only pennies and nickels, or choose 1 dime and add that to any way of making $k - 10$ with pennies, nickels, and dimes.

Continuing in this way, we can use $q_k$ for the number of ways to make $k$ using pennies, nickels, dimes, and quarters; $h_k$ for the number of ways uses pennies through half dollars, and $a_k$ as before for the number of ways of using any of the coins.
We get the following equations:

\[ q_k = \begin{cases} 
  d_k & \text{if } 0 \leq k \leq 24, \\
  d_k + q_{k-25} & \text{if } k \geq 25; 
\end{cases} \]

\[ h_k = \begin{cases} 
  q_k & \text{if } 0 \leq k \leq 49, \\
  q_k + h_{k-50} & \text{if } k \geq 50; 
\end{cases} \]

\[ a_k = \begin{cases} 
  h_k & \text{if } 0 \leq k \leq 99, \\
  h_k + a_{k-100} & \text{if } k \geq 100; 
\end{cases} \]

This does not give a closed form for \( a_k \).

Using these formulas, it’s easy to make a table and compute any particular \( a_k \) that we’re interested in.
Penny and nickel values:

<table>
<thead>
<tr>
<th>Value</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pn</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
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<td>7</td>
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<td>9</td>
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<td>2</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>12</td>
<td>15</td>
<td>18</td>
<td>21</td>
<td>24</td>
<td>27</td>
</tr>
<tr>
<td>4k</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>13</td>
<td>17</td>
<td>21</td>
<td>25</td>
<td>29</td>
<td>33</td>
</tr>
</tbody>
</table>

10k = P_{10} + n_{5}

- 10k = P_{10} + 0 (no nickel)
- 10k = P_{10} + 1 (at least 1 nickel)

\[ a_{15} = P_{15} + d_{15,10} \]
\[ a_{20} = P_{20} + d_{20,10} \]
\[ a_{25} = P_{25} + a_{20} \]
\[ a_{30} = P_{30} + a_{25} \]
In fact, we can make this calculation simpler because the denominations of all the coins except pennies are multiples of 5, but that doesn’t apply in the more general setting where we’re combining weights or lengths or some other quantities. (See problem 10 from Section 2.6.3.)

\[ \text{# of ways to write } n \text{ as a sum of positive integers} = \binom{k+4}{4} + 45 \binom{k+3}{4} + 52 \binom{k+2}{4} + 2 \binom{k+1}{4} \]

If we had coins of every denomination 1, 2, 3, ....

gen function \( \frac{1}{(1-x)(1-x^2)(1-x^3) \cdots} \) and coeff of \( x^n \) is the number of partitions of \( n \). (\# of ways to write \( n \) as a sum of positive integers)