Project questionnaire, instructions for submitting homework, etc.

- Project questionnaire responses due at start of today’s class! I will get team assignments to you by the end of this week.

- Instructions for logging in to gradescope.com, submitting homework, etc.
  - Cristian wrote these up and they’re now posted on both the course web page and Moodle.

- No responses to my question about web page vs. Moodle...
Combinatorics is the study of “arrangements”. The book mentions several flavors of combinatorics:

- **Enumerative:** Count arrangements satisfying various constraints
- **Existential:** Determine whether certain arrangements can exist
- **Constructive:** Algorithms for constructing special arrangements.

There are other ways of subdividing the field, too, based on the kinds of techniques used (e.g., analytic combinatorics), the particular kinds of problems studied, or the applications (e.g., to representation theory).

In this course, we’ll focus on enumerative combinatorics: counting things.
Some basics

- Sum rule: If $S_1, \ldots, S_n$ are pairwise disjoint sets with $|S_i| = n_i$, then the number of ways to select an object from $S = \bigcup_{i=1}^{n} S_i$ is $\sum_{i=1}^{n} n_i$.

- Product rule: $S_1, \ldots, S_n$ as above. The number of ways to select an object from $S_1$, followed by an object from $S_2$, \ldots, followed by an object from $S_n$ is $\prod_{i=1}^{n} n_i$.

  - So the number of ways to order $n$ objects is $n(n-1)(n-2)\cdots2\cdot1 = n!$
  
  - Number of ways to select an ordered list of $k$ objects from a set of $n$ objects is $P(n,k) = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1)$

  - Number of ways to select a subset (unordered) of $k$ objects from a set of $n$ objects is $\frac{P(n,k)}{k!} = \frac{n!}{(n-k)!k!} = C(n,k) = \binom{n}{k}$ "$n$ choose $k$"
Binomial coefficients

I’m going to assume you’re familiar with binomial coefficients, know the binomial theorem (combinatorial proof?), and some of the basic identities involving binomial coefficients. You’ve probably seen most of these:

- \( \binom{n}{k} = \binom{n}{n-k} \)
- \( \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \) \text{ Pascal’s Triangle}
- \[ \sum_{k=0}^{n} \binom{n}{k} = 2^n \] \( \binom{n}{k} \) is the number of subsets of order \( k \) in a set of order \( n \)
- \[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} = (-1+1)^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases} \] 

But there are lots more. The book gives a few in Section 2.2.

**Binomial Th:**

\[(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

**Proof:** \( n \text{ terms} \)

\[(x+y)(x+y)(x+y) \cdots (x+y) \]

To get a term \( k \cdot x^k y^{n-k} \)

choose \( k \) of the \( n \) factors to take the \( x \) from, and take the \( y \) from the remaining factors.

So the number of terms \( k \cdot x^k y^{n-k} \)

is the number of ways to choose the \( k \) factors from which to take the \( x \) term. That number of ways is \( \binom{n}{k} \)

}\}
$X$ a set of $n$ elements. How many subsets? (What is $|\mathcal{P}(X)|$?)

$x_1 \ x_2 \ x_3 \ \ldots \ x_n$

subsets \quad \{0, 1, 0, \ldots, 1\} \quad \text{1 in the $i$th spot}

if $x_i$ is in the subset

and 0 otherwise

$\# \text{of subsets} = \# \text{of n-vectors of 0s and 1s}$

2 choices for each slot

Product rule says $\# \text{of vectors is} \ 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$

$n$ times
Multinomial coefficients

- Number of ways to divide \( n \) objects up into a set of \( a \) and a set of \( b = n - a \):
  - It’s enough to pick the subset of size \( a \), and we know there are \( \binom{n}{a} \) ways to do that. And \( \binom{n}{a} = \binom{n}{n-a} \)

- Number of ways to divide \( n \) objects up into 3 subsets of size \( a, b, \) and \( c \) (so \( c = n - a - b \)):
  - First select a subset of size \( a \): \( \binom{n}{a} \) ways to do that.
  - Then, from the remaining \( n - a \) objects, select a subset of size \( b \): \( \binom{n-a}{b} \) ways to do that.
  - Last subset is what’s left.
So number of ways to partition $n$ objects into subsets of size $a$, $b$, and $c = n-a-b$ is

$$\binom{n}{a}\binom{n-a}{b} = \frac{n!}{(n-a)!a! (n-a-b)!b!}$$

$$= \frac{n!}{a!b!(n-a-b)!} = \frac{n!}{a!b!c!} = \binom{n}{a,b,c}$$

We call this a **multinomial coefficient**. These play the same role in $(x+y+z)^n$ as the binomial coefficients do in $(x+y)^n$. (Prove it combinatorially the same way.)

$$(x+y+z)(x+y+z)\ldots (x+y+z) = \sum \binom{n}{a,b,c} x^a y^b z^c$$

where $n = b_1 + b_2 + \ldots + b_m$
An application of multinomial coefficients

A multiset (sometimes called a bag) is a collection that can have more than one copy of each of its elements (unlike a standard set).

- So the set of letters in UMASS is \{U, M, A, S\}, but as a multiset we would have \{U, M, A, S, S\} with two instances of S as an element.

Suppose we have a multiset with a total of \(n\) elements, consisting of \(k_1\) identical copies of one object, \(k_2\) copies of another, and so on, up to \(k_m\) copies of the last one. (So \(n = k_1 + \cdots + k_m\).) How many different ways are there of ordering the \(n\) elements?

- \(n!\) ways to order the \(n\) objects as if they were all distinct. But can reorder the \(k_1\) objects of the first type without change, so divide by \(k_1!\); and reorder the \(k_2\) objects of the second type without change, so divide by \(k_2!\), \ldots, giving

\[
\frac{n!}{k_1!k_2!\cdots k_m!} = \binom{n}{k_1, k_2, \ldots, k_m}
\]

Choose \(k_1\) slots for objects of 1st type, \(k_2\) slots for objects of 2nd type, \ldots.
Problem 9a: Compute the number of 3-letter and 4-letter sequences that can be formed by the letters in "Bug Tussle, TX", ignoring differences in case.

3-letter sequences:

- We have 1 B, 2 Us, 1 G, 2 Ts, 2 Ss, 1 L, 1 E, and 1 X

- First figure out how many 3-element multisets we get

- Then, for each multiset, figure out how many different ways to order the elements of that multiset.
2 kinds of multisets to consider

- 3 distinct letters \( w, x, y \) 
  \[ \binom{8}{3} \text{ such sets} \]
  \[ 3! \text{ ways to order such a set} \]
  \[ 3! \cdot \binom{8}{3} = \frac{8! \cdot 3!}{5! \cdot 3!} = 56 \cdot 6 \\
  \]

- 2 instances of one letter, 1 instance of another \( w, w, x \)
  1 of the letters occurring twice \( U, T, S \)
  1 of the letters other than the one we already chose
  \[ 3 \cdot 7 = 21 \]
  \[ 3 \text{ ways to order each } w, w, x \]
  \[ 56 \cdot 6 + 21 \cdot 3 \]

3-letter strings
4. Letter strings

2 different letters, each twice \( \text{w} \text{x} \text{w} \text{xy} \)  
- 2 letters from the 3 that occur twice \( \binom{3}{2} \) ways to do that  
- order \( \binom{4}{2} \) in each case

1 letter twice, 2 letters each once \( \text{w} \text{x} \text{y} \text{xy} \)  
- \( \binom{4}{2} \) order each multiset \( \binom{2}{1,1} \)  
- \( 4! \) orders if I could tell the w's apart

4 different letters \( \text{w} \text{x} \text{y} \text{z} \)  
- \( \binom{8}{4} \) multisets, \( 4! \) ways to order each one  
- \( \frac{4!}{2} \) orders when I can't tell the w's apart

Book has another example with more repetition
Used sum rule and product rule in fairly complicated ways

Pigeonhole principle: Several applications in book.

How closely can we approximate an irrational \( \alpha \) by rational \( \frac{p}{q} \) with \( q \) bounded: \( 1 \leq q \leq Q \)?

Dirichlet Approx Thm: There exists rational \( \frac{p}{q} \) with \( 1 \leq q \leq Q \) and

\[
\left| \alpha - \frac{p}{q} \right| < \frac{1}{q(Q+1)}
\]

Proof: Divide \([0,1]\) into \( Q+1 \) equal subintervals

\([0, \frac{1}{Q+1}], [\frac{1}{Q+1}, \frac{2}{Q+1}], \ldots\)

Write \( \delta \) for fractional part of \( \alpha \)

\[
\lfloor \alpha \rfloor + \delta = \alpha
\]

\( \text{largest integer} \leq \alpha \)
The $2k+2$ numbers $0, 3x^2, \ldots, 3Qx^2$, 1 must have two of them in the same subinterval.

Each has form $nx-s$ for integers $r, s$, $0 \leq r \leq Q$

Have $r_1, r_2, s_1, s_2$ with

$$| (r_1x-s_1)-(r_2x-s_2) | < \frac{c}{Q+1}$$

Take $r_1 < r_2$ and $q = r_2 - r_1$, then $1 \leq q \leq Q$

Take $p = s_2 - s_1$. Then

$$|q \cdot x - p| < \frac{1}{Q+1} \quad \text{so} \quad |x - \frac{p}{q}| < \frac{c}{q(Q+1)}.$$