$S_k = I + A + A^2 + \ldots + A^k$

$\text{ecc}(v_i)$ is smallest $k \geq 0$ where the $i$th row of $S_k$ is all nonzero

Trees: connected, acyclic graph

A leaf is a vertex of degree 1

A forest is a graph where each connected component is a tree (a collection of one or more trees)

Theorem 1.12: A connected graph of order $n$ is a tree if and only if its size is $n-1$

Pf: $\leq$ Induction on $n$

If every vertex has degree $\geq 2$

$2 \cdot \text{size} = \sum_{v \in V} \deg(v) \geq 2n$

$\Rightarrow$ size $\leq n$ but size $= n-kn$

delete vertex $v$ of degree 1. $G-v$ has size $n-1$ & order $n-2$

By induction $G-v$ is a tree

vertex $v$ not in a cycle since $\deg(v) = 1$

So $G$ has no cycles

Assume $G$ has a cycle, delete edge on a cycle, $G$ still connected

$n$ vertices, $n-1$ edges, not connected!
Induction

Base cases \( n = 1, 2 \) easy.
Suppose true if \( n < k \) consider tree of order \( k \). Want to show size is \( k - 1 \).

Remove edge: get connected components \( T_1 \) and \( T_2 \).
\( T_1 \) and \( T_2 \) are trees of order \( < k \).
So, \( T_1 \) has size \( k_1 - 1 \) and \( T_2 \) has size \( k_2 - 1 \).
\( G \) has size \( k_1 - 1 + k_2 - 1 + 1 = k_1 + k_2 - 1 \).

Theorem 1.14. A tree of order \( n \geq 2 \) has at least \( 2 \) leaves.

Pf. true if \( n = 2 \).
Suppose true for \( n < k \). Suppose we have tree \( G \) of order \( k \geq 3 \).
If every edge is incident with a leaf, done.
If not, choose an edge where neither edge vertex is a leaf and delete the edge. \( G - e \) is a forest with \( 2 \) components \( T_1, T_2 \).
\( u \in T_1, v \in T_2 \) orders \( k_1, k_2 \) respectively \( k_1, k_2 \geq 2 \).