Section 2.1

6. Construct a truth table for each of the following statements:

(a) \( P \lor \neg Q \)

\[
\begin{array}{c|c|c}
P & Q & P \lor \neg Q \\
\hline
T & T & F \\
T & F & T \\
F & T & F \\
F & F & T \\
\end{array}
\]

(b) \( \neg (P \land Q) \)

\[
\begin{array}{c|c|c|c}
P & Q & P \land Q & \neg (P \land Q) \\
\hline
T & T & F & T \\
T & F & T & T \\
F & T & F & F \\
F & F & T & T \\
\end{array}
\]

(c) \( \neg P \lor \neg Q \)

\[
\begin{array}{c|c|c|c|c}
P & Q & \neg P & \neg Q & \neg P \lor \neg Q \\
\hline
T & T & F & F & F \\
T & F & F & T & T \\
F & T & T & F & T \\
F & F & T & T & T \\
\end{array}
\]

(d) \( \neg P \land \neg Q \)

\[
\begin{array}{c|c|c|c|c}
P & Q & \neg P & \neg Q & \neg P \land \neg Q \\
\hline
T & T & F & F & F \\
T & F & F & T & F \\
F & T & T & F & F \\
F & F & T & T & T \\
\end{array}
\]

8. Suppose each of the following statements is true:
   
   – Laura is in the seventh grade.
– Laura got an A on the mathematics test or Sarah got an A on the mathematics test.
– If Sarah got an A on the mathematics test, then Laura is not in the seventh grade.

If possible, determine the truth value of each of the following statements. Carefully explain your reasoning.

(a) Laura got an A on the mathematics test.

We know that Laura is in seventh grade and that at least one of Laura and Sarah got an A on the test. If Sarah got an A, Laura is not in seventh grade; however, we know Laura is in seventh grade so it must be false that Sarah got an A on the test. Therefore Laura got an A on the test.

(b) Sarah got an A on the mathematics test.

This is false, as noted in the first part. If Sarah got an A, Laura would not be in seventh grade but we know Laura is in seventh grade.

(c) Either Laura or Sarah did not get an A on the mathematics test.

This is true, since we know that Sarah did not get an A.

12.a For statements $P$, $Q$, and $R$:

(a) Show that $[(P \to Q) \land P] \to Q$ is a tautology. (Note: In symbolic logic, this is an important logical argument form called modus ponens.)

This is easy to see with a truth table:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \to Q$</th>
<th>$(P \to Q) \land P$</th>
<th>$[(P \to Q) \land P] \to Q$</th>
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The first line of the truth table is clear. In the second line, $P \to Q$ is false, making $(P \to Q) \land P$ false as well. But the last implication is true when the hypothesis is false. In the third and fourth line, $(P \to Q) \land P$ is also false, because $P$ is false. That makes $[(P \to Q) \land P] \to Q$ true regardless of the truth value of $Q$.

(b) Show that $[(P \to Q) \land (Q \to R)] \to (P \to R)$ is a tautology. Note: In symbolic logic, this is an important logical argument from called syllogism.
Here's the truth table:

<table>
<thead>
<tr>
<th></th>
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<th>R</th>
<th>P → Q</th>
<th>Q → R</th>
<th>(P → Q) ∧ (Q → R)</th>
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<th>(P → Q) ∧ (P → Q) → (P → R)</th>
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Here’s another approach that doesn’t require filling in the truth table.

The top-level implication is true except when the hypothesis is true and the conclusion is false. To show the given implication is a tautology, we need to show that this case can never arise for the given implication.

The hypothesis is [(P → Q) ∧ (Q → R)] and the conclusion is P → R. We assume that the hypothesis is true and the conclusion is false. For the conclusion to be false, we must have P true and R false.

So consider the hypothesis when P is true and R is false and recall that we want to show that the hypothesis cannot be true. For this to happen, we must have both P → Q and Q → R true. Then, since both P and P → Q are true, we know that Q is true (by part (a), modus ponens). Applying modus ponens again using the fact that Q and Q → R are true, we conclude that R is true. But we already know that R is false, so this is a contradiction. Hence, it is impossible for [(P → Q) ∧ (Q → R)] to be true when P is true and R is false, so when P → R is false.

14. Suppose that P and Q are true statements, that U and V are false statements, and that W is a statement and it is not known if W is true or false. Which of the following statements are true, which are false, and for which statements is it not possible to determine if it true or false? Justify your conclusions.

(a) (P ∨ Q) ∨ (U ∧ W)

Since P is true, P ∨ Q is true, and that makes (P ∨ Q) ∨ (U ∧ W) true.

(b) P ∧ (Q → W)

We don’t know whether Q → W is true; it is false if W is false. So we can’t determine whether the given statement is true or false even though we know P is true.

(f) (¬P ∨ ¬U) ∧ (Q ∨ ¬V)
This is true. The statements $\neg U$ and $\neg V$ are both true, so both parts of the conjunction are true.

Section 2.2

3.f Write a useful negation of the following statement. Do not leave a negation as a prefix of a statement.

If you graduate from college, then you will get a job or you will go graduate school.

You will graduate from college and neither get a job nor go to graduate school.

5. Use truth table to prove each of the distributive laws from Theorem 2.8

(a) $P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R)$

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<th>$P$</th>
<th>$Q$</th>
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<th>$Q \land R$</th>
<th>$P \lor (Q \land R)$</th>
<th>$P \lor Q$</th>
<th>$P \lor R$</th>
<th>$(P \lor Q) \land (P \lor R)$</th>
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(b) $P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$

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9. Use previously proven logical equivalences to prove each of the following:

(a) $[\neg P \rightarrow (Q \land \neg Q)] \equiv P$

We know that $Q \land \neg Q$ is false, and $\neg P \rightarrow F$ is equivalent to $\neg(\neg P) \lor F$, which is just $P \lor F$. But $P \lor F$ is true exactly when $P$ is true, so this is equivalent to $P$. 

4
(b) \((P \leftrightarrow Q) \equiv (\neg P \lor Q) \land (\neg Q \lor P)\)

We know that \(P \leftrightarrow Q \equiv (P \rightarrow Q) \land (Q \rightarrow P)\). And \(R \rightarrow S \equiv \neg R \lor S\), so \((P \rightarrow Q) \land (Q \rightarrow P) \equiv (\neg P \lor Q) \land (\neg Q \lor P)\).

11. Let \(a\), \(b\), and \(c\) be integers. Consider the following conditional statement:

If \(a\) divides \(bc\), then \(a\) divides \(b\) or \(a\) divides \(c\).

Which of the following statements have the same meaning as this conditional statement and which ones are negations of this conditional statement? (This is not asking which statements are true and which are false.)

(a) If \(a\) divides \(b\) or \(a\) divides \(c\), then \(a\) divides \(bc\).

Let’s think of the original statement as \(P \rightarrow (Q \lor R)\), where \(P\) is the statement that \(a\) divides \(bc\), \(Q\) is the statement that \(a\) divides \(b\), and \(R\) is the statement that \(a\) divides \(c\).

So this is \((Q \lor R) \rightarrow P\), the converse of the original conditional statement. It’s not equivalent to the original statement, and it’s not the negation (which would be \(P \land \neg(Q \lor R)\), or \(P \land (\neg Q \land \neg R)\)). We saw in class that the converse of a conditional statement is not equivalent to the original statement, and it’s pretty clearly not the negation because the negation would be true only when \(P\) is true and \(Q \lor R\) is false, whereas this statement is true, for instance, when \(Q \lor R\) is true and \(P\) is true.

(b) If \(a\) does not divide \(b\) or \(a\) does not divide \(c\), then \(a\) does not divide \(bc\).

This is \((\neg Q \lor \neg R) \rightarrow \neg P\), which is almost the contrapositive of the original statement. But \(\neg Q \lor \neg R\) is not the negation of \((Q \lor R)\). The conditional statement “If \(a\) does not divide \(b\) and \(a\) does not divide \(c\), then \(a\) does not divide \(bc\)” would have the same meaning as the original statement. This statement, for instance, is true when \(Q\) and \(R\) are false and \(P\) is true.

It’s also not the negation of the original statement, since both this and the original are true when \(P\), \(Q\), and \(R\) are all true.

(c) \(a\) divides \(bc\), \(a\) does not divide \(b\) and \(a\) does not divide \(c\).

This is \(P \land \neg Q \land \neg R\). So it is true exactly when \(P\) is true, \(Q\) is false, and \(R\) is false. It’s clearly not equivalent to the original statement, which is true when \(P\) and \(Q\) are true and \(R\) is false, for instance.
The original statement is false only when $P$ is true and both $Q$ and $R$ are false (so that $Q \lor R$ is false). But this says that the negation of the original statement is true only in that case, and therefore this statement is equivalent to the negation of the original statement.

(d) If $a$ divides $bc$ and $a$ does not divide $c$, then $a$ divides $b$.

This is $(P \land \neg R) \rightarrow Q$. It is equivalent to the given statement, since the only way it can be false is for $P$ to be true and both $R$ and $Q$ to be false.