Section 6.2:

8. Let $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined by $g(m, n) = (2m, m - n)$.

b. Determine all the preimages of $(0, 0)$. That is, find all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that $g(m, n) = (0, 0)$.

The only preimage is $(0, 0)$. If $g(m, n) = (0, 0)$, then $2m = 0$, so $m = 0$, and $m - n = 0$, so $n = m$.

– Determine the set of all preimages of $(1, 1)$.

The set of preimages of $(1, 1)$ is $\emptyset$ since the first coordinate of $g(m, n)$ is always even.

Section 6.3:

3. For each of the following functions, determine if the function is an injection and determine if the function is a surjection. Justify all conclusions.

– (c.) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$, for all $x \in \mathbb{R}$.

This function is both injective and surjective. If $g(a) = g(b)$, then $a^3 = b^3$. But then $a = \sqrt[3]{a^3} = \sqrt[3]{b^3} = b$, so $g$ is injective. And clearly $g(\sqrt[3]{y}) = y$, so $g$ is surjective.

e. $k : \mathbb{R} \to \mathbb{R}$ defined by $k(x) = e^{-x^2}$, for all $x \in \mathbb{R}$.

This function is not injective because, e.g., $k(-1) = k(1)$.

And it’s not surjective since, for any $x \in \mathbb{R}$, $x^2 \geq 0$, so $e^{-x^2} \geq 1$. But then $e^{-x^2} = \frac{1}{e^{x^2}} \leq 1$. So $k(x) \leq 1$ for all $x$.

5. Let $s : \mathbb{N} \to \mathbb{N}$, where, for each $n \in \mathbb{N}$, $s(n)$ is sum of the distinct natural number divisors of $n$. Is $s$ and injection? Is $s$ a surjection? Justify your conclusions.

The sum of divisors function is not an injection, since, e.g., $s(6) = s(11) = 12$. And it’s not a surjection, since $s(n) \neq 2$ for all $n \in \mathbb{N}$: every $n$ divides itself and 1 divides $n$, so if $1 \neq n$, $s(n) \geq n + 1$. But $s(1) = 1$, so the preimage of 2 under $s$ is empty. but every

8a. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $f(m, n) = 2m + n$. Is the function $f$ an injection? Is the function $f$ a surjection? Justify your conclusions.

It’s not an injection, since, for example, $f(1, 1) = 3 = f(0, 3)$. It is, however, a surjection. To see this, let $n \in \mathbb{Z}$. Then clearly $f(0, n) = 2 \cdot 0 + n = n$. 

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17. a. There are several problems with the way the proof is written: it doesn’t define \( f \) so it’s not really self-contained. (This is minor.) The argument from the system of equations to show that \( a = c \) and \( b = d \) is unclear—the argument starts by simply listing 4 equations, the last of which is \( a = c \). A better proof would state the two equations in the system, and then explain that the equation \( 3a = 3c \) is obtained by adding those, concluding that \( a = c \) from the equation \( 3a = 3c \). It would also explicitly note that we have proved \((a, b) = (c, d)\).

b. This is a good proof, but it would be better to start out by saying why finding such an ordered pair would show that \( f \) is a surjection.

Section 6.4:

1. In our definition of the composition of two functions, \( f \) and \( g \), we required that the domain of \( g \) be equal to the codomain of \( f \). However, it is sometimes possible to form the compositie function \( g \circ f \) even though \( \text{domain}(g) \neq \text{codomain}(f) \). For example, let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x^2 + 1 \), and let \( g : \mathbb{R} - \{0\} \to \mathbb{R} \) be defined by \( g(x) = 1/x \).

a. Is it possible to determine \((g \circ f)(x)\) for all \( x \in \mathbb{R} \)? Explain.

Yes. Since \( f(x) \in \text{domain}(g) \) for all \( x \in \mathbb{R} \), \((g \circ f)(x) = g(x^2 + 1) = 1/(x^2 + 1)\) for all \( x \in \mathbb{R} \).

b. In general, let \( f : A \to T \) and \( g : B \to C \). Find a condition on the domain of \( g \) (other than \( B = T \)) that results in a meaningful definition of the composite function \( g \circ f : A \to C \).

If \( T \subseteq B \), we can define the composite function as usual, with \( g \circ f = \{(a, c) \in A \times C \mid \exists t \in T \text{ with } (a, t) \in f \text{ and } (t, c) \in g\} \). (This is the same as defining \( (g \circ f)(a) = g(f(a)) \).)

7. For each of the following, give an example of functions \( f : A \to B \) and \( g : B \to C \) that satisfy the stated conditions, or explain why no such example exists.

c. The function \( g \) is a surjection, but the function \( g \circ f \) is not a surjection.

One example is \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(s) = s^2 \) and \( g : \mathbb{R} \to \mathbb{R} \) defined by \( g(x) = x \). Then \( g \) is clearly a surjection (the identity function is always a bijection), but the image of \( g \circ f \) is only the nonnegative real numbers.

d. The function \( g \) is an injection, but the function \( g \circ f \) is not an injection.

We can use the same \( f \) and \( g \) as in part (c). We know that \( g \) is injective, but the composition sends \(-1\) and \(1\) to the same value so \( g \circ f \) is not injective.

e. The function \( f \) is not a surjection, but the function \( g \circ f \) is a surjection.
Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = e^x \) and let \( g : \mathbb{R}^+ \to \mathbb{R} \) (where \( \mathbb{R}^+ \) is the book’s notation for the positive real numbers) be defined by \( g(x) = \ln(x) \). Then \( g \circ f \) is a surjection, but \( f \) is not a surjection.

Section 6.5:

9. Prove that, if \( f : A \to B \) is a bijection, then \( f^{-1} : B \to A \) is also a bijection.

Since \( f : A \to B \) is a bijection, we know that the set \( f^{-1} = \{ (b, a) \mid (a, b) \in f \} \subseteq B \times A \) is a function. We show that it’s both injective and surjective.

To see \( f^{-1} \) is injective, suppose that \( b_1, b_2 \in B \) with \( f^{-1}(b_1) = f^{-1}(b_2) \). Then (since we know that \( f \circ f^{-1} = I_B \)), we have \( b_1 = f(f^{-1}(b_1)) = f(f^{-1}(b_2)) = b_2 \). So \( b_1 = b_2 \), and we have shown that \( f^{-1} \) is injective.

To see that \( f^{-1} \) is surjective, let \( a \in A \). Then \( f^{-1}(f(a)) = a \) (since \( f^{-1} \circ f = I_Z \)), so \( f^{-1} \) is surjective.

Extra Credit:

1. Prove that the function \( f : X \to Y \) is injective if and only if it satisfies the following condition: For any set \( T \) and functions \( g : T \to X \) and \( h : T \to X \), \( f \circ g = f \circ h \) implies \( g = h \).

Suppose that \( f \) is injective and \( f \circ g = f \circ h \). Let \( t \) be any element of \( T \), and let \( x_g = g(t) \) and \( x_h = h(t) \). Since \( f \circ g = f \circ h \), we have \( f(g(t)) = f(h(t)) \), so \( f(x_g) = f(x_h) \). But \( f \) is injective, so this can only happen if \( x_g = x_h \). Thus, \( g(t) = h(t) \). Since this is true for any element \( t \in T \), we have \( g = h \).

Conversely, suppose that, for any set \( T \) and functions \( g : T \to X \) and \( h : T \to X \), \( f \circ g = f \circ h \) implies \( g = h \). Suppose \( x_1, x_2 \in X \) with \( f(x_1) = f(x_2) \). Take \( T = \{ t \} \) and define \( g(t) = x_1 \) and \( h(t) = x_2 \). Then, since \( f(x_1) = f(x_2) \), \( f \circ g = f \circ h \) and our hypothesis says that \( g = h \). But that implies that \( g(t) = h(t) \), so \( x_1 = x_2 \). This shows that \( f \) is injective.

2. Prove that the function \( f : X \to Y \) is surjective if and only if it satisfies the following condition: For any set \( Z \), and any functions \( g : Y \to Z \) and \( h : Y \to Z \), \( g \circ f = h \circ f \) implies \( g = h \). [Hint: To prove \( f \) is surjective, take \( Z = \{ z_1, z_2 \} \) with \( g(y) = z_1 \) for all \( y \in Y \) and \( h(y) = z_1 \) for all \( y \in f(X) \), but \( h(y) = z_2 \) for all \( y \notin f(X) \).]

Suppose that \( f \) is surjective and \( g : Y \to Z \) and \( h : Y \to Z \) are functions such that \( g \circ f = h \circ f \). We want to show that \( g = h \), which means that \( g(y) = h(y) \) for all \( y \in Y \). Consider \( y \in Y \). Since \( f \) is surjective, there exists \( x \in X \) such that
\( f(x) = y. \) Then, since \( g \circ f = h \circ f, \) we have \( g(f(x)) = h(f(x)), \) which says \( g(y) = h(y). \) So \( g = h. \)

Conversely, suppose that \( f \) satisfies the given condition. We want to show that \( f \) is surjective, which means \( Y = f(X). \) Following the hint, take \( Z = \{z_1, z_2\} \) with \( g(y) = z_1 \) for all \( y \in Y \) and \( h(y) = z_1 \) for all \( y \in f(X), \) but \( h(y) = z_2 \) for all \( y \notin f(X). \) For all \( y \in \text{Im}(f) = f(X), \) \( g(y) = h(y), \) so \( g \circ f = h \circ f. \) Suppose that \( f(X) \subsetneq Y, \) and choose some \( y \notin f(X). \) Then \( g(y) = z_1, \) but \( h(y) = z_2, \) so \( g \neq h. \) This contradicts our assumption, so it must be the case that \( f(X) = Y \) and \( f \) is surjective.