Math 300.3—Spring 2019
Homework due March 28

Section 6.2:

8. Let \( g: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \) be defined by \( g(m, n) = (2m, m - n) \).

b. Determine all the preimages of \((0, 0)\). That is, find all \((m, n) \in \mathbb{Z} \times \mathbb{Z}\) such that \( g(m, n) = (0, 0) \).

The only preimage is \((0, 0)\). If \( g(m, n) = (0, 0) \), then \( 2m = 0 \), so \( m = 0 \), and \( m - n = 0 \), so \( n = m \).

− Determine the set of all preimages of \((1, 1)\).

The set of preimages of \((1, 1)\) is \(\emptyset\) since the first coordinate of \( g(m, n) \) is always even.

Section 6.3:

3. For each of the following functions, determine if the function is an injection and determine if the function is a surjection. Justify all conclusions.

− (c.) \( g: \mathbb{R} \to \mathbb{R} \) defined by \( g(x) = x^3 \), for all \( x \in \mathbb{R} \).

This function is both injective and surjective. If \( g(a) = g(b) \), then \( a^3 = b^3 \). But then \( a = \sqrt[3]{a^3} = \sqrt[3]{b^3} = b \), so \( g \) is injective. And clearly \( g(\sqrt[3]{y}) = y \), so \( g \) is surjective.

e. \( k: \mathbb{R} \to \mathbb{R} \) defined by \( k(x) = e^{-x^2} \), for all \( x \in \mathbb{R} \).

This function is not injective because, e.g., \( k(-1) = k(1) \).
And it’s not surjective since, for any \( x \in \mathbb{R} \), \( x^2 \geq 0 \), so \( e^{-x^2} \geq 1 \). But then \( e^{-x^2} = \frac{1}{e^{x^2}} \leq 1 \). So \( k(x) \leq 1 \) for all \( x \).

5. Let \( s: \mathbb{N} \to \mathbb{N} \), where, for each \( n \in \mathbb{N} \), \( s(n) \) is sum of the distinct natural number divisors of \( n \). Is \( s \) and injection? Is \( s \) a surjection? Justify your conclusions.

The sum of divisors function is not an injection, since, e.g., \( s(6) = s(11) = 12 \). And it’s not a surjection, since \( s(n) \neq 2 \) for all \( n \in \mathbb{N} \): every \( n \) divides itself and 1 divides \( n \), so if \( 1 \neq n \), \( s(n) \geq n + 1 \). But \( s(1) = 1 \), so the preimage of 2 under \( s \) is empty, but every

8a. Let \( f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) be defined by \( f(m, n) = 2m + n \). Is the function \( f \) an injection? Is the function \( f \) a surjection? Justify your conclusions.

It’s not an injection, since, for example, \( f(1, 1) = 3 = f(0, 3) \). It is, however, a surjection. To see this, let \( n \in \mathbb{Z} \). Then clearly \( f(0, n) = 2 \cdot 0 + n = n \).
17. a. There are several problems with the way the proof is written: it doesn’t define $f$ so it’s not really self-contained. (This is minor.) The argument from the system of equations to show that $a = c$ and $b = d$ is unclear—the argument starts by simply listing 4 equations, the last of which is $a = c$. A better proof would state the two equations in the system, and then explain that the equation $3a = 3c$ is obtained by adding those, concluding that $a = c$ from the equation $3a = 3c$. It would also explicitly note that we have proved $(a, b) = (c, d)$.

b. This is a good proof, but it would be better to start out by saying why finding such an ordered pair would show that $f$ is a surjection.

Section 6.4:

1. In our definition of the composition of two functions, $f$ and $g$, we required that the domain of $g$ be equal to the codomain of $f$. However, it is sometimes possible to form the composite function $g \circ f$ even though $\text{domain}(g) \neq \text{codomain}(f)$. For example, let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$, and let $g: \mathbb{R} - \{0\} \to \mathbb{R}$ be defined by $g(x) = 1/x$.

   a. Is it possible to determine $(g \circ f)(x)$ for all $x \in \mathbb{R}$? Explain.

      Yes. Since $f(x) \in \text{domain}(g)$ for all $x \in \mathbb{R}$, $(g \circ f)(x) = g(x^2 + 1) = 1/(x^2 + 1)$ for all $x \in \mathbb{R}$.

   b. In general, let $f: A \to T$ and $g: B \to C$. Find a condition on the domain of $g$ (other than $B = T$) that results in a meaningful definition of the composite function $g \circ f: A \to C$.

      If $T \subseteq B$, we can define the composite function as usual, with $g \circ f = \{(a, c) \in A \times C \mid \exists t \in T \text{ with } (a, t) \in f \text{ and } (t, c) \in g\}$. (This is the same as defining $(g \circ f)(a) = g(f(a))$.)

7. For each of the following, give an example of functions $f: A \to B$ and $g: B \to C$ that satisfy the stated conditions, or explain why no such example exists.

   c. The function $g$ is a surjection, but the function $g \circ f$ is not a surjection.

      One example is $f: \mathbb{R} \to \mathbb{R}$ defined by $f(s) = s^2$ and $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x$. Then $g$ is clearly a surjection (the identity function is always a bijection), but the image of $g \circ f$ is only the nonnegative real numbers.

   d. The function $g$ is an injection, but the function $g \circ f$ is not an injection.

      We can use the same $f$ and $g$ as in part (c). We know that $g$ is injective, but the composition sends $-1$ and $1$ to the same value so $g \circ f$ is not injective.

   e. The function $f$ is not a surjection, but the function $g \circ f$ is a surjection.
Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$ and let $g: \mathbb{R}^+ \to \mathbb{R}$ (where $\mathbb{R}^+$ is the book's notation for the positive real numbers) be defined by $g(x) = \ln(x)$. Then $g \circ f$ is a surjection, but $f$ is not a surjection.

10. Use the ideas from Exercise (9) to prove theorem 6.21. Let $A$, $B$, and $C$ be nonempty sets and let $f: A \to B$ and $g: B \to C$.

(a) If $g \circ f: A \to C$ is an injection, then $f$ is an injection.

We prove the contrapositive. Suppose that $f$ is not an injection. Then there exist $a_1, a_2 \in A$ with $a_1 \neq a_2$ but $f(a_1) = f(a_2)$. Then $g \circ f(a_1) = g(f(a_1)) = g(f(a_2)) = g \circ f(a_2)$. Since $a_1 \neq a_2$, this shows that $g \circ f$ is not an injection.

(b) If $g \circ f$ is a surjection, then $g$ is a surjection.

Since $g \circ f$ is a surjection, we know that, for all $c \in C$, there exists an $a \in A$ such that $g \circ f(a) = c$. But $g \circ f(a) = g(f(a))$, so, taking $b = f(a)$, we know that for all $c \in C$, there exists a $b \in B$ such that $c = g(b)$. This shows that $g$ is a surjection.

Section 6.5:

9. Prove that, if $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ is also a bijection.

Since $f: A \to B$ is a bijection, we know that the set $f^{-1} = \{(b, a) \mid (a, b) \in f\} \subseteq B \times A$ is a function. We show that it’s both injective and surjective.

To see $f^{-1}$ is injective, suppose that $b_1, b_2 \in B$ with $f^{-1}(b_1) = f^{-1}(b_2)$. Then (since we know that $f \circ f^{-1} = I_B$), we have $b_1 = f(f^{-1}(b_1)) = f(f^{-1}(b_2)) = b_2$. So $b_1 = b_2$, and we have shown that $f^{-1}$ is injective.

To see that $f^{-1}$ is surjective, let $a \in A$. Then $f^{-1}(f(a)) = a$ (since $f^{-1} \circ f = I_B$), so $f^{-1}$ is surjective.

Extra Credit:

1. Prove that the function $f: X \to Y$ is injective if and only if it satisfies the following condition: For any set $T$ and functions $g: T \to X$ and $h: T \to X$, $f \circ g = f \circ h$ implies $g = h$.

Suppose that $f$ is injective and $f \circ g = f \circ h$. Let $t$ be any element of $T$, and let $x_g = g(t)$ and $x_h = h(t)$. Since $f \circ g = f \circ h$, we have $f(g(t)) = f(h(t))$, so $f(x_g) = f(x_h)$. But $f$ is injective, so this can only happen if $x_g = x_h$. Thus, $g(t) = h(t)$. Since this is true for any element $t \in T$, we have $g = h$. 


Conversely, suppose that, for any set $T$ and functions $g: T \rightarrow X$ and $h: T \rightarrow X$, $f \circ g = f \circ h$ implies $g = h$. Suppose $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$. Take $T = \{t\}$ and define $g(t) = x_1$ and $h(t) = x_2$. Then, since $f(x_1) = f(x_2)$, $f \circ g = f \circ h$ and our hypothesis says that $g = h$. But that implies that $g(t) = h(t)$, so $x_1 = x_2$. This shows that $f$ is injective.

2. Prove that the function $f: X \rightarrow Y$ is surjective if and only if it satisfies the following condition: For any set $Z$, and any functions $g: Y \rightarrow Z$ and $h: Y \rightarrow Z$, $g \circ f = h \circ f$ implies $g = h$. [Hint: To prove $f$ is surjective, take $Z = \{z_1, z_2\}$ with $g(y) = z_1$ for all $y \in Y$ and $h(y) = z_1$ for all $y \in f(X)$, but $h(y) = z_2$ for all $y \notin f(X)$.

Suppose that $f$ is surjective and $g: Y \rightarrow Z$ and $h: Y \rightarrow Z$ are functions such that $g \circ f = h \circ f$. We want to show that $g = h$, which means that $g(y) = h(y)$ for all $y \in Y$. Consider $y \in Y$. Since $f$ is surjective, there exists $x \in X$ such that $f(x) = y$. Then, since $g \circ f = h \circ f$, we have $g(f(x)) = h(f(x))$, which says $g(y) = h(y)$. So $g = h$.

Conversely, suppose that $f$ satisfies the given condition. We want to show that $f$ is surjective, which means $Y = f(X)$. Following the hint, take $Z = \{z_1, z_2\}$ with $g(y) = z_1$ for all $y \in Y$ and $h(y) = z_1$ for all $y \in f(X)$, but $h(y) = z_2$ for all $y \notin f(X)$. For all $y \in \text{Im}(f) = f(X)$, $g(y) = h(y)$, so $g \circ f = h \circ f$. Suppose that $f(X) \subsetneq Y$, and choose some $y \notin f(X)$. Then $g(y) = z_1$, but $h(y) = z_2$, so $g \neq h$. This contradicts our assumption, so it must be the case that $f(X) = Y$ and $f$ is surjective.