Section 4.2:

11. Prove that for each odd natural number \(n\) with \(n \geq 3\),

\[
\left(1 + \frac{1}{2}\right) \left(1 = \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \ldots \left(1 + \frac{(-1)^n}{n}\right) = 1.
\]

We reindex using \(n = 2m + 1\) so that we’re proving a statement for all natural numbers \(m\), rather than for just odd natural numbers greater than or equal to 3. So let \(P(m)\) be the statement that

\[
\left(1 + \frac{1}{2}\right) \left(1 = \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \ldots \left(1 + \frac{(-1)^{2m+1}}{2m + 1}\right) = 1.
\]

Then \(P(1)\) is true since \(\frac{3}{2} \cdot \frac{2}{4} = 1\).

We need to show that \(P(k) \rightarrow P(k+1)\) for each natural number \(k\). This means that

\[
\left(1 + \frac{1}{2}\right) \left(1 = \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \ldots \left(1 + \frac{(-1)^{2k+1}}{2k + 1}\right) = 1. \quad (1)
\]

Then we need to prove \(P(k+1)\), which says

\[
\left(1 + \frac{1}{2}\right) \left(1 = \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \ldots \left(1 + \frac{(-1)^{2(k+1)+1}}{2(k+1) + 1}\right) = 1. \quad (2)
\]

Multiplying both sides of Equation 1 by \(\left(1 + \frac{1}{2k+2}\right) \left(1 - \frac{1}{2k+3}\right)\)
gives us the left side of Equation 2 on the left and

\[
1 \cdot \left(1 + \frac{1}{2k + 2}\right) \left(1 - \frac{1}{2k + 3}\right) = \frac{2k + 3}{2k + 2} \cdot \frac{2k + 2}{2k + 3} = 1
\]
on the right. This proves \(P(k) \rightarrow P(k+1)\). \(\square\)

16. (a) Prove that if \(n \in \mathbb{N}\), there exists an odd natural number \(m\) and a nonnegative integer \(k\) such that \(n = 2^k m\).

We will prove this using the second principle of induction.

The base case is \(n = 1\). But \(1 = 2^0 \cdot 1\) and 1 is odd, so the result holds for \(n = 1\).

Suppose the result holds for \(n = 1, 2, \ldots, j\). We will prove that this implies it holds for \(n = j + 1\).

We know that \(j + 1\) is either odd or even. If \(j + 1\) is odd, then \(j + 1 = 2^0 \cdot (j + 1)\) and the result holds. If \(j + 1\) is even,
then \( j + 1 = 2a \) for some integer \( a \) with \( 1 \leq a < j + 1 \). So the result holds for \( a \) by assumption, and we have \( a = 2^{k_1}m_1 \), with \( m_1 \) odd.

Then \( j + 1 = 2a = 2(2^{k_1}m_1) = 2^{k_1+1}m_1 \). Taking \( k = k_1 + 1 \) and \( m = m_1 \), we have \( j + 1 = 2^k m \) with \( m \) odd, so the result holds for \( j + 1 \) when it holds for \( j \).

By the second principle of mathematical induction, the result holds for all positive integers.

(b) For each \( n \in \mathbb{N} \), prove that there is only one way to write \( n \) in the form described in part (a). To do this, assume that \( n = 2^k m \) and \( n = 2^q p \), where \( m \) and \( n \) are odd natural numbers and \( k \) and \( q \) are nonnegative integers. Then prove that \( k = q \) and \( m = p \).

Taking the hint, assume \( n = 2^k m \) and \( n = 2^q p \), where \( m \) and \( n \) are odd natural numbers and \( k \) and \( q \) are nonnegative integers. If \( k > q \), then we have \( 2^{k-q} m = p \), making the left side even and the right side odd. This is impossible. A similar argument shows that \( k < q \) is impossible. So we must have \( k = q \). But then our assumption says \( 2^k m = 2^q p \), which implies that \( m = p \). So we have proved \( k = q \) and \( m = p \). Hence, there is only one way to write \( n \) in the form described in part (a).

17. Evaluate the proofs, as usual. I will not reproduce the proofs here.

(a) For each natural number \( n \) with \( n \geq 2 \), \( 2^n > 1 + n \).

This is a presentation of an induction proof that might be ok for experts, who will recognize that it’s an induction proof early and perhaps don’t need to have the predicate defined, the base step noted, etc. But it’s better to say something like:

Proof. We will prove this proposition by induction. For each natural number \( n \geq 2 \), let \( P(n) \) be the predicate

\[
2^n > 1 + n.
\]

The base case is \( n = 2 \), where we have \( 4 = 2^2 > 1 + 2 \). So \( P(2) \) is true.

For the induction step, assume that \( P(k) \) is true with \( k \geq 2 \). This says that

\[
2^k > 1 + k. \quad (3)
\]

We need to prove that \( 2^{k+1} > 1 + (k + 1) \). Multiplying Equation 3 by 2 gives

\[
2 \cdot 2^k > 2(1 + k)
\]

\[
2^{k+1} > 2 + 2k > 2 + k
\]

\[
2^{k+1} > 1 + (k + 1).
\]
This proves $P(k) \rightarrow P(k+1)$, so we have finished the induction proof.

(b) Each natural number greater than or equal to 6 can be written as the sum of natural numbers, each of which is a 2 or a 5.

This is almost a correct proof, but it uses $k \geq 10$ and never proves $P(10)$. (But $10 = 2 \cdot 5$, so $P(10)$ is true.) The proof could also be improved by stating more carefully how the goal of proving $P(k+1)$ is to be achieved, i.e., by finding nonnegative integers $u, v$ such that $k + 1 = 2u + 5v$.

Also, you could do this with fewer special base cases if you looked at $(k + 1) - 2$ in the induction step, instead of $k+1-5$. And you could prove a stronger proposition, since 4 and 5 can also be written as $2x + 5y$ for $x$ and $y$ nonnegative integers.

Extra problems:

• Use induction to prove that $a^m \cdot a^n = a^{m+n}$ for all $n \in \mathbb{N}$. (You’re proving this for all $m \in \mathbb{N}$. Recall the discussion from class about using an “arbitrary” element to prove universally qualified statements. So your induction is on $n$.)

The base case is $n = 1$. Then $a^n = a$, so

$$a^m \cdot a^n = a^m \cdot a = \underbrace{a \cdot a \cdots a}_m \cdot a = \underbrace{a \cdot a \cdots a}_m = a^{m+1}.$$ 

So the assertion holds for $n = 1$. In fact, we have proved that, for all $m \in \mathbb{N}$, $a^m \cdot a = a^{m+1}$.

For the induction step, assume that $a^m \cdot a^k = a^{m+k}$. We want to use this to prove that $a^m \cdot a^{k+1} = a^{m+k+1}$. We have

$$a^m \cdot a^{k+1} = \underbrace{a \cdot a \cdots a}_{k+1 \text{ times}}$$

by the associative property of multiplication,

$$= \underbrace{(a \cdot a \cdots a)}_k \cdot a$$

from the inductive hypothesis,

$$= a^{m+k} \cdot a$$

$$= a^{m+k+1}$$
from proving the base case for all \( m \).

- Use induction to prove that a convex \( n \)-gon contains \( \frac{1}{2} n(n - 3) \) diagonals. (A *diagonal* of a convex \( n \)-gon is the line segment joining two nonadjacent vertices, i.e., vertices that don’t share an edge.)

  This can be proved without using induction, but we’ll do it by induction. (A *diagonal* of a convex \( n \)-gon is the line segment joining two nonadjacent vertices, i.e., vertices that don’t share an edge.)

  In order to get a convex \( n \)-gon, we must have \( n \geq 3 \). So the base case is \( n = 3 \). A 3-gon is a triangle, which has no diagonals. For \( n = 3 \), \( \frac{1}{2} n(n - 3) = 0 \), so the base case holds.

  So assume that a convex \( k \)-gon has \( \frac{1}{2} k(k - 3) \) diagonals and consider a convex \( (k + 1) \)-gon \( P \), for some \( k \geq 3 \). Fix a vertex \( x \) and let \( y \) and \( z \) be its adjacent vertices, so \( xy \) and \( xz \) are two edges of the \((k + 1)\)-gon. There are \( k + 1 - 3 \) vertices other than \( x \), \( y \), and \( z \), so \( P \) has \( k - 2 \) diagonals having \( x \) as one of the endpoints. Note that there is also a diagonal connecting \( y \) and \( z \).

  Consider the \( k \)-gon \( Q \) that is obtained by removing \( x \) and the edges \( xy \) and \( xz \), and making the diagonal \( yz \) into an edge. This is a \( k \)-gon (and it’s easy to check that it’s convex, but I’ll leave that to you), so it has \( \frac{1}{2} k(k - 3) \) diagonals. But any two vertices of \( Q \) that are nonadjacent were nonadjacent vertices of \( P \), so all the diagonals of \( Q \) are diagonals of \( P \). The only diagonals of \( P \) that are not diagonals of \( Q \) are the ones involving \( x \) and \( yz \) (since that’s an edge of \( Q \)). So \( P \) has \( k - 2 + 1 \) more diagonals than \( Q \).

  Since

  \[
  k-1+\frac{1}{2}k(k-3) = \frac{2k - 2 - k(k - 3)}{2} = \frac{k^2 - k - 2}{2} = \frac{1}{2}k(k+1)(k-2),
  \]

  we have proved that a convex \((k+1)\)-gon has \( \frac{1}{2}k(k+1)((k+1)-3) \) diagonals from the assumption that a convex \( k \)-gon has \( \frac{1}{2}k(k-3) \) diagonals. This completes the induction proof. \( \square \)