Section 2.4:

9. An integer $m$ is said to have the divides property provided that for all integers $a$ and $b$, if $m$ divides $ab$, then $m$ divides $a$ or $m$ divides $b$. [Note that “divides property” is not a standard term.]

(a) Using the symbols for quantifiers, write what it means to say that the integer $m$ has the divides property.

You don’t have to use symbols for implication, etc., just for quantifiers. So something like this is fine:
\[(\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}) \text{ if } m \text{ divides } ab, \text{ then } m \text{ divides } a \text{ or } m \text{ divides } b.\]

But you could also say:
\[\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, (m \mid ab) \rightarrow ((m \mid a) \lor (m \mid b)).\]

(b) Using the symbols for quantifiers, write what it means to say that the integer $m$ does not have the divides property.

Again, you don’t have to put the statement fully into symbolic form. But here’s an acceptable answer:
\[\exists a \in \mathbb{Z}, \exists b \in \mathbb{Z}, (m \mid ab) \land (m \nmid a) \land (m \nmid b).\]

11. In calculus, we define a function $f$ to be continuous at a real number $a$ provided that, for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Complete each of the following sentences using the appropriate symbols for quantifiers.

(a) A function $f$ is continuous at the real number $a$ provided that . . .
\[\forall \varepsilon > 0, (\forall \delta > 0), (|x - a| > \delta) \rightarrow (|f(x) - f(a)| < \varepsilon).\]

(b) A function $f$ is not continuous at the real number $a$ provided that . . .
\[\exists \varepsilon > 0, (\forall \delta > 0), (\exists x), (|x - a| < \delta) \land (|f(x) - f(a)| \geq \varepsilon).\]

Note that the existential quantifier on $x$ comes from the hidden universal quantifier in the conditional $|x - a| < \delta \rightarrow |(f(x) - f(a))| < \varepsilon$.

Section 3.1:

9. Let $a$ and $b$ be integers. Prove that if $a \equiv 7 \pmod{8}$ and $b \equiv 3 \pmod{8}$, then

(a) $a + b \equiv 2 \pmod{8}$
Assuming that \( a \equiv 7 \pmod{8} \) and \( b \equiv 3 \pmod{8} \), there exist integers \( m \) and \( n \) such that \( a = 8m + 7 \) and \( b = 8n + 3 \).

Then,

\[
a + b = (8m + 7) + (8n + 3) \\
= 8m + 8n + 8 + 2 \\
= 8(m + n + 1) + 2.
\]

This proves that \( a + b \equiv 2 \pmod{8} \).

(b) \( a \cdot b \equiv 5 \pmod{8} \)

Assuming that \( a \equiv 7 \pmod{8} \) and \( b \equiv 3 \pmod{8} \), there exist integers \( m \) and \( n \) such that \( a = 8m + 7 \) and \( b = 8n + 3 \).

Then,

\[
a \cdot b = (8m + 7)(8n + 3) \\
= 64mn + 24m + 56n + 21 \\
= 8(8mn + 3m + 7n + 2) + 5.
\]

This proves \( a \cdot b \equiv 5 \pmod{8} \).

10. Determine if each of the following propositions is true or false. Justify each conclusion.

(a) For all integers \( a \) and \( b \), if \( ab \equiv 0 \pmod{6} \), then \( a \equiv 0 \pmod{6} \) or \( b \equiv 0 \pmod{6} \).

This statement is false. A counterexample is \( a = 3 \) and \( b = 2 \). With these values, \( ab = 6 \equiv 0 \pmod{6} \), but \( a \not\equiv 0 \pmod{6} \) and \( b \not\equiv 0 \pmod{6} \).

(b) For each integer \( a \), if \( a \equiv 2 \pmod{8} \), then \( a^2 \equiv 4 \pmod{8} \).

This is true. Here’s a proof.

Assuming \( a \equiv 2 \pmod{8} \), we know that \( 8 \mid a - 2 \) and hence there exists an integer \( n \) such that \( 8n = a - 2 \). So \( a = 8n + 2 \) and \( (a^2 = 64n^2 + 32n + 4) \). So \( a^2 - 4 = 8(8n^2 + 4n) \).

Thus, \( 8 \mid (a^2 - 4) \), so \( a^2 \equiv 4 \pmod{8} \).

(c) For each integer \( a \), if \( a^2 \equiv 4 \pmod{8} \), then \( a \equiv 2 \pmod{8} \).

This statement is false. Consider \( a = 6 \). Then \( a^2 = 36 = 8 \cdot 4 + 4 \), so \( a^2 \equiv 4 \pmod{8} \). But \( 6 \equiv 2 \) is not divisible by \( 8 \), so \( 6 \not\equiv 2 \pmod{8} \).

11.c Let \( n \) be a natural number. Prove that, for all integers \( a \), \( b \), and \( c \), if \( a \equiv b \pmod{n} \) and \( b \equiv c \pmod{n} \), then \( a \equiv c \pmod{n} \). This is called the **transitive property** of congruence modulo \( n \).

Assume \( a \equiv b \pmod{n} \) and \( b \equiv c \pmod{n} \). So there exist integers \( k \) and \( \ell \) such that \( kn = (a - b) \) and \( \ell n = (b - c) \). Adding these equations gives \( kn - \ell n = a - b + b - c = a - c \). So \( a - c \) is divisible by \( n \) and \( a \equiv c \pmod{n} \).
19. If the proposition is false, find the error in the proof and provide a counterexample. If the proposition is true and the proof is incorrect, explain why and write a correct proof using the writing guidelines in the book. If the proposition is true and the proof is correct, decide whether the proof is well written. If not, revise the proof by writing it according to the guidelines in the text.

(a) Proposition. If \( m \) is an even integer, then \( (5m + 4) \) is an even integer.

Proof. We see that \( 5m + 4 = 10n + 4 = 2(5n + 2) \). Therefore \( (5m + 4) \) is an even integer.

This proof is not well written. The assumptions are not stated, there is no indication of what will be proved, and the definition of even is not explicitly mentioned. It might also be good to display the equations, though they’re simple enough here that it’s not crucial. Here’s a better proof.

Proof. Let \( m \) be an even integer. This means that there exists an integer \( k \) such that \( m = 2k \). Then

\[
5m + 4 = 5(2k) + 4 = 10k + 4 = 2(5k + 2).
\]

Since \( k \) is an integer, \( 5k + 2 \) is an integer, and we have shown that \( 5m + 4 \) is 2 times an integer. This means that \( 5m + 4 \) is an even integer.

(b) Proposition. For all real numbers \( x \) and \( y \), if \( x \neq y \), \( x > 0 \), and \( y > 0 \), then \( \frac{x}{y} + \frac{y}{x} > 2 \).

Proof. Since \( x \) and \( y \) are positive real numbers, \( xy \) is positive and we can multiply both sides of the inequality by \( xy \) to obtain

\[
\left( \frac{x}{y} + \frac{y}{x} \right) \cdot xy > 2 \cdot xy
\]

\[
x^2 + y^2 > 2xy.
\]

By combining all terms on the left side of the inequality, we see that \( x^2 - 2xy + y^2 > 0 \), and then by factoring the left side, we obtain \((x - y)^2 > 0 \). Since \( x \neq y \), \((x - y) \neq 0 \) and so \((x - y)^2 > 0 \). This proves that if \( x \neq y \), \( x > 0 \), and \( y > 0 \), then \( \frac{x}{y} + \frac{y}{x} > 2 \).

The proposition is true, but the proof is not valid. It starts from the conclusion of the proposition and proves the hypotheses. So this is a proof of the converse of the proposition, not the given proposition. It’s also not very well written, since it doesn’t state the assumptions, define \( x \) and \( y \), etc. Here’s a better version.
Proof. Let $x$ and $y$ be positive real numbers and assume $x \neq y$. We need to show that $\frac{x}{y} + \frac{y}{x} > 2$. Since $x \neq y$, we know $x - y \neq 0$ so we have

\[
(x - y)^2 > 0
\]
\[
x^2 - 2xy - y^2 > 0
\]
\[
x^2 + y^2 > 2xy.
\]

Since $x, y > 0$, we know that $xy \neq 0$ and we can divide both sides of the last inequality by $xy$ to get

\[
\frac{x^2 + y^2}{xy} > 2
\]
\[
\frac{x^2}{xy} + \frac{y^2}{xy} > 2
\]
\[
\frac{x}{y} + \frac{y}{x} > 2,
\]

as claimed. This completes the proof. \qed

(c) Proposition. For all integers $a$, $b$, and $c$, if $a \mid (bc)$ then $a \mid b$ or $a \mid c$.

Proof. We assume that $a$, $b$, and $c$ are integers and that $a$ divides $bc$. So there exists an integer $k$ such that $bc = ka$. We now factor $k$ as $k = mn$, where $m$ and $n$ are integers. We then see that $bc = mna$. This means that $b = ma$ or $c = na$ and hence, $a \mid b$ or $a \mid c$.

This proposition is false. A counterexample is $a = 4$, $b = 2$, and $c = 2$. Here, $bc = 4$, which is certainly divisible by 4, but 2 is not divisible by 4.

Section 3.2:

11. Prove that for each integer $a$, if $a^2 - 1$ is even, then 4 divides $a^2 - 1$.

Suppose that $a$ is an integer and $a^2 - 1$ is even. So $a^2$ is odd, which implies that $a$ is odd. (It’s ok to just say that at this point, but you could prove it by, for example, proving the contrapositive: $a$ even $\rightarrow$ $a^2$ even.) This means that there is an integer $k$ with $a = 2k + 1$. But then $a^2 - 1 = 4k^2 + 4k + 1 - 1 = 4(k^2 + k)$. Since $k^2 + k$ must be an integer, this proves that 4 divides $a^2 - 1$.

Section 3.3:

2.c Is the following statement true or false? Justify your answer. “For all integers $a$ and $b$, if $a$ is even and $b$ is odd, then 4 does not divide $(a^2 + 2b^2)$.
This statement is true. We can prove it by contradiction, as follows. Assume that $a$ and $b$ are integers, with $a$ even and $b$ odd, and that 4 does divide $a^2 + 2b^2$. Since $a = 2k$ for some integer $k$ and $b = 2\ell + 1$ for some integer $\ell$, there must be an integer $n$ such that

$$4n = (2k)^2 + 2(2\ell + 1)^2$$

$$= 4k^2 + 2(4\ell^2 + 4\ell + 1)$$

$$= 4k^2 + 8\ell^2 + 4\ell + 2$$

$$= 4(k^2 + 2\ell^2 + \ell) + 2.$$  

Subtracting $4(k^2 + 2\ell^2 + \ell)$ from both sides, we see that 2 is an integer multiple of 2. This is false, so our assumption that 4 divides $a^2 + 2b^2$ must be false. 

6. Are the following statements true or false? Justify each conclusion.

(a) For every pair of real numbers $x$ and $y$, if $x + y$ is irrational, then $x$ is irrational and $y$ is irrational.

This is false. A counterexample is $x = \sqrt{2}$, $y = 0$. We have $x + y = x$ irrational, but $y$ rational.

(b) For every pair of real numbers $x$ and $y$, if $x + y$ is irrational, then $x$ is irrational or $y$ is irrational.

This is true. The contrapositive of the conditional is “($x$ is rational and $y$ is rational) $\Rightarrow$ $x + y$ is rational”. This is true ($\mathbb{Q}$ is closed under addition).

20.a Evaluate the following proof as usual.

**Proposition.** For each real number $x$, if $x$ is irrational and $m$ is an integer, then $mx$ is irrational.

**Proof.** We assume that $x$ is a real number and is irrational. This means that for all integers $a$ and $b$ with $b \neq 0$, $x \neq \frac{a}{b}$. Hence, we may conclude that $mx \neq \frac{ma}{b}$, and therefore $mx$ is irrational.

This is false. A counterexample is $x = \sqrt{2}$ and $m = 0$, in which case $mx = 0$, which is certainly rational.

If we change the hypotheses to include the assumption that $m \neq 0$, the revised proposition would be true. But the given proof is not a good one even for this version of the proposition. The tricky point is the argument that, if $mx \neq \frac{ma}{b}$ for all integers $a, b$ with $b \neq 0$, then $mx$ must be irrational. But the definition of irrational requires us to consider all fractions (with nonzero denominator), not just ones with numerators divisible by $m$. It’s true that, by considering all fractions $\frac{ma}{b}$, as $a$ and $b$ run over the integers with $b \neq 0$, we actually do get all fractions
with integer numerators and denominators—to get any \( \frac{c}{d} \), we can take \( a = c \) and \( b = md \), so \( \frac{ma}{b} = \frac{c}{d} \). But this has to be explained in the proof.

Another approach would be to prove the revised version of the proposition is by contradiction:

**Proof.** Assume there exists an irrational real number \( x \) and an integer \( m \neq 0 \) such that \( mx \) is rational. Then there are integers \( a \) and \( b \), \( b \neq 0 \), such that \( mx = \frac{a}{b} \). But then \( x = \frac{a}{mb} \). Since \( a \) is an integer and \( mb \) is an integer not equal to 0, this would imply that \( x \) is rational. This contradicts our hypothesis that \( x \) is irrational and we conclude that \( mx \) is also irrational. \( \square \)