Math 300 Exam 1 Solutions  
February 28, 2019

This exam has 7 questions and 4 pages, with 100 total points. Make sure your copy is complete and put 
your name on all 4 pages so I can identify your pages if they get separated. If you need more room, you 
can write on the backs of the exam pages (but, please, indicate clearly that you’re doing that—otherwise I 
may not look at the backs of the pages).

Show all your work and justify your answers (except in cases where you’re explicitly told 
you don’t need to. You will not get credit for answers not supported by work.
You may not use any books, notes, or electronic devices during the exam.

1. Give precise and complete definitions of the following terms:
   (a) (5 points) A statement is . . .

   **Solution:** a declarative sentence that is unambiguously either true or false.

   (b) (5 points) Two statements are logically equivalent if and only if

   **Solution:** they have the same truth value for all possible combinations of the truth values of 
   the variables appearing in the two statements.

   (c) (5 points) A compound statement is a tautology if and only if . . .

   **Solution:** it is true for all possible combinations of truth values of its component statements.

   (d) (5 points) The Second Principle of Mathematical Induction says . . .

   **Solution:** for $M \in \mathbb{Z}$, if $T$ is a subset of $\mathbb{Z}$ satisfying the two conditions

   (i.) $M \in T$, and

   (ii.) For every $k \in \mathbb{Z}$ with $k \geq M$, if $M, M + 1, \ldots, k \in T$, then $k + 1 \in T$,
   then $T = \{ n \in \mathbb{Z} | n \geq M \}$.

2. (10 points) Write the truth table for the statement $(P \land Q) \rightarrow \neg R$. (You don’t have to show any work except the truth table.)

   **Solution:**

   \[
   \begin{array}{c|c|c|c|c|c}
   P & Q & R & P \land Q & \neg R & (P \land Q) \rightarrow \neg R \\
   \hline
   T & T & T & T & F & F \\
   T & T & F & T & T & T \\
   T & F & T & F & F & T \\
   T & F & F & F & T & T \\
   F & T & T & F & F & T \\
   F & T & F & F & T & T \\
   F & F & T & F & F & T \\
   F & F & F & F & T & T \\
   \end{array}
   \]
3. We discussed the universal and existential quantifiers in class. Often, we want to say that there is exactly one element \( x \) in a set \( X \) for which the predicate \( P(x) \) true. For this, we have the unique existential quantifier \( \exists! \) and we write \( \exists! x \in X, P(x) \). One way to define it in terms of the ordinary existential and universal quantifiers is:

\[
\exists! x \in X, P(x) \iff \exists x \in X, (P(x) \land (\forall y \in X, (P(y) \rightarrow x = y)))
\]

(a) (7 points) Carefully translate the right side of the above biconditional into English.

**Solution:** There is an \( x \in X \) such that \( P(x) \) is true and, for all \( y \in X \), if \( P(y) \) is true, then \( x = y \).

(b) (13 points) Use the logical expression above to write a logical expression for \( \neg \exists! x \in X, P(x) \). Simplify your expression as much as you can.

**Solution:**

\[
\neg (\exists! x \in X, P(x)) \equiv \neg (\exists x \in X, (P(x) \land (\forall y \in X, (P(y) \rightarrow x = y))))
\]

\[
\equiv \forall x \in X, \neg (P(x) \land (\forall y \in X, (P(y) \rightarrow x = y)))
\]

\[
\equiv \forall x \in X, \neg P(x) \lor \neg (\forall y \in X, (P(y) \rightarrow x = y))
\]

\[
\equiv \forall x \in X, \neg P(x) \lor \exists y \in X, \neg (P(y) \rightarrow x = y)
\]

\[
\equiv \forall x \in X, \neg P(x) \lor \exists y \in X, (P(y) \land x \neq y)
\]

This says that, for all \( x \in X \), either \( P(x) \) is false or there’s some other element \( y \) with \( P(y) \) true. So one possibility is that \( P(x) \) is false for all \( x \). Otherwise, there’s at least one \( x \) for which \( P(x) \) is true, but there’s a \( y \) not equal to that \( x \) for which \( P(y) \) is also true.
4. (15 points) Use induction to prove that, for every natural number \(n\),
\[
\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\cdots\left(1 + \frac{1}{n}\right) = n + 1
\]

**Solution:** Let \(P(n)\) be the statement that
\[
\prod_{i=1}^{n} \left(1 + \frac{1}{i}\right) = n + 1.
\]

We will use induction to prove that this statement holds for all natural numbers \(n\).

The base case is \(P(1)\). We have
\[
\prod_{i=1}^{1} \left(1 + \frac{1}{i}\right) = 1 + 1 = 2.
\]
We also have \(n + 1 = 2\). So \(P(1)\) is true.

For the induction step, we must prove that, for any natural number \(k\), \(P(k) \rightarrow P(k + 1)\). So we choose a \(k \in \mathbb{N}\). To prove the implication, we assume that \(P(k)\) is true and use that to try to prove \(P(k + 1)\) is true (without using anything about \(k\) except that it’s a natural number). Thus, we start from the assumption that
\[
\prod_{i=1}^{k} \left(1 + \frac{1}{i}\right) = k + 1,
\]
and we want to prove that
\[
\prod_{i=1}^{k+1} \left(1 + \frac{1}{i}\right) = (k + 1) + 1.
\]

So consider
\[
\prod_{i=1}^{k+1} \left(1 + \frac{1}{i}\right) = \left(\prod_{i=1}^{k} \left(1 + \frac{1}{i}\right)\right) \left(1 + \frac{1}{k+1}\right)
\]
By the assumption about \(P(k)\), we know the right-hand side is
\[
= (k + 1) \left(1 + \frac{1}{k+1}\right)
\]
\[
= (k + 1) + \frac{k + 1}{k+1}
\]
\[
= k + 1 + 1
\]
\[
= k + 2.
\]
So we have proved that \(P(k) \rightarrow P(k + 1)\) for all natural numbers \(k\). By the Principle of Mathematical Induction, this tells us that \(P(n)\) is true for all natural numbers \(n\). 

\[\square\]
5. (20 points) Prove the following statement: For all \( x \in \mathbb{R} \), there exists \( y \in \mathbb{R} \) such that \( x + y = xy \) if and only if \( x \neq 1 \).

**Solution:** Let \( x \) be a real number.

We will first show by contradiction that, if there is a \( y \in \mathbb{R} \) with \( x + y = xy \), then \( x \neq 1 \). So assume that there is a \( y \in \mathbb{R} \) with \( x + y = xy \) and that \( x = 1 \). Since \( x = 1 \), this says \( 1 + y = y \). Subtracting \( y \) from both sides gives \( 1 = 0 \), which is false. This means that our assumption is false. Thus, if there exists a \( y \in \mathbb{R} \) such that \( x + y = xy \), it must be true that \( x \neq 1 \).

For the converse, suppose \( x \neq 1 \) and consider the equation \( x + y = xy \). Subtracting \( x \) from both sides gives \( x = y(x - 1) \) and we solve to get \( y = \frac{x}{x-1} \). Since \( x \neq 1 \), this is a real number satisfying \( x + y = xy \).

6. Let \( P \) be the statement “Today is Tuesday”; \( Q \) be the statement “I am in Italy”; \( R \) be the statement “I will go to a great restaurant”; and \( S \) be the statement “It is a sunny day”.

(a) (5 points) Give a logical formula equivalent to the statement “If I am in Italy then it is both a sunny day and I will go to a great restaurant.”

**Solution:** \( Q \rightarrow (S \land R) \).

(b) (5 points) Express this statement in English: \( \neg Q \rightarrow \neg R \).

**Solution:** If I am not in Italy, then I will not go to a great restaurant.

7. (5 points) Give an example of a true conditional statement with a false contrapositive, or explain why no such example can exist.

**Solution:** The conditional statement \( P \rightarrow Q \) is equivalent to its contrapositive, \( \neg Q \rightarrow \neg P \). If two statements are equivalent, it’s not possible for one to be true when the other is false. So no such example can exist.