My plan is to use the Gradescope “online assignment” feature to administer the exam, but I have not yet set up the exam this way in Gradescope and Gradescope lists this feature as “beta”. So I may encounter some issues and need to change some of the things I’m currently planning to do. (Watch the course web page and your email for announcements of any changes.) In this format, most of your answers will have to be uploaded as a pdf file, presumably scanned from something you’ve written on paper, just as with the homework assignments you’ve turned in.

If all goes as currently planned, the exam will become available to you at 8:00 am EDT (Amherst time) on Tuesday, April 14. Once you start the exam, you will have to complete it within 2.5 hours, and you must complete the exam (including uploading your answers) by 8:00 am EDT on Wednesday, April 15. You may use the textbook and your notes from the course. You may not use any other sources (books, the internet, other people, etc.). I will ask you to sign (by uploading a scan) an honor pledge. You can send me email with questions about the exam, but I will not be available for the whole 24 hours, so you can’t count on getting a reply from me before you have to submit your answers.

The exam covers the material in the text in Chapters 5, 6, and 7. So I won’t ask you a question that’s entirely on material from earlier chapters. But the material in this course is extremely cumulative. So you still need to know and be able to use the material that was covered on the first exam. For instance, there won’t be a question that’s just on induction proofs, but I expect you to be able to use induction to prove things about, e.g., a particular equivalence relation.

You’re responsible for the material in the textbook and everything we’ve done in class, including the Zoom sessions. (Recall that you can view the recordings of the Zoom sessions, but you have to register to view each recording and I have to approve it, so if you wait to register until you’re taking the exam, you may not be approved in time to actually view the recording.) Here’s a bit more detail on the topics (though the fact that something is listed here doesn’t mean it will necessarily be on the exam and the fact that something isn’t mentioned here doesn’t mean it won’t be on the exam).

The questions on the exam will be similar to the ones on the first exam, but I won’t ask you for definitions or other things that can be straightforwardly taken from the textbook. So I may ask you to show that a certain function is a bijection, but I won’t ask you for the definition of a bijection. I might give you a proof and ask you to critique it, as you have done for some homework problems.

Chapter 5: This chapter covers the basic definitions of set, element, subset, containment, etc. You should know these definitions, and the basic operations
on sets, such as intersection, union, and complement. The most important note to make about proof techniques is that showing two sets are equal is usually done by separately showing that the first is a subset of the second and that the second is a subset of the first. Of course, the most basic way to show that one set is a subset of another is to show that, for all elements $x$ of the first set, $x$ is also an element of the second; and proving a universally quantified statement usually starts by choosing an “anonymous” element of the set.

You also need to know the properties of the operations, including De Morgan’s Laws, the distributive properties of union and intersection, etc.

An important construction in this chapter is the Cartesian product of two sets.

The chapter also talks about operations involving indexed families of sets. I think most of Section 5.5 is pretty straightforward and you shouldn’t worry about it too much, but notice the introduction of the “big” union and intersection symbols that function like the large Σ we use for adding indexed collections of numbers.

**Chapter 6:** Chapter 6 defines functions and establishes properties of certain classes of functions. Presumably you already knew about functions; the first section of the chapter just reviews that definition and establishes some terminology (domain, codomain, range, etc.). Note that the definition of a function $f : A \to B$ includes the domain and codomain; for two functions to be equal, they must have the same domain and codomain. In class, we talked about formally defining the function $f : A \to B$ a special kind of subset of the Cartesian product $A \times B$; the book doesn’t discuss this until Section 6.5, but I think it’s important for you to have this in mind as the way we make precise sense out of the somewhat vague idea of “a rule that associates” an element of the codomain with each element of the domain.

Section 6.2 is mostly about examples of functions. Section 6.3 defines injections, surjections, and bijections (which you’ve probably encountered previously under the names “one-to-one functions”, “onto functions”, and “one-to-one and onto functions”). You need to know the definitions and be able to prove that particular functions are injective or surjective or bijective. Section 6.4 introduces composition of functions, which is also something you’ve encountered before. The definition here is the same as the one you’ve used in, e.g., calculus (but make sure you understand how composition works if we define a function as a subset of the Cartesian product). The book states some theorems about the composition of injections, etc. (though some of the proofs are left to the exercises). You should know these results and be able to prove things like them (or provide counterexamples for the versions which aren’t true). You should also know about the special identity function $I_A$ from a set $A$ to itself that maps each element to itself. (As a subset of the Cartesian product $A \times A$, it’s just the diagonal, the set $\{(a, a) \mid a \in A\}$.)

Section 6.5 is about inverse functions. The main result here is that a function $f : A \to B$ has an inverse function $f^{-1} : B \to A$ if and only if $f$ is a bijection.
(in which case \(f^{-1}\) is also a bijection), but I think the book’s presentation is a little confusing. This is where the book introduces the idea of a function as a set of ordered pairs. Given a function \(f: A \rightarrow B\), not necessarily bijective, the book defines a subset of \(B \times A\) it calls “the inverse of \(f\)” and denotes by \(f^{-1}\). The key point is that this will just be a relation and not a function unless \(f\) is bijective. (I think it’s confusing to call this “the inverse of \(f\)” when it’s not a function; it’s the inverse relation in a sense that’s not exactly the same as the inverse of a function, and the section is called “Inverse Functions”, so I’m not surprised if some of you had a little trouble sorting this out.)

In class, I gave a definition of an invertible function and the inverse of a function in terms of the identity functions on sets. If \(f: A \rightarrow B\) and \(g: B \rightarrow A\) satisfy the conditions that \(g \circ f = I_A\) and \(f \circ g = I_B\), we say that \(f\) and \(g\) are invertible and that \(g = f^{-1}\) and \(f = g^{-1}\). Such a \(g\) exists exactly when \(f\) is bijective and, in this case, we already know \(g\) is a function, and we saw that it has to be the same as the book’s definition: \(f^{-1}(b)\) is the unique \(a\) (which has to exist and be unique because \(f\) is bijective) that has \(f(a) = b\). (So the difference here is that the book is willing to call something “the inverse of \(f\)” even when that thing is not a function and \(f\) is not invertible.) The book’s Corollary 6.28 is essentially the condition about the compositions being the identity functions in my definition of invertible.

Section 6.6 talks about images and preimages of subsets of the domain and codomain under a function \(f\). This is the part where I said we were extending the notation, so that if \(f: A \rightarrow B\) is a function, we can talk about \(f(a)\) for an element \(a \in A\) and \(f(X)\) for a set \(X \subseteq A\). And we use the \(f^{-1}(Y)\) notation for the preimage of a set \(Y \subseteq B\) (though usually not for an element \(b \in B\) unless \(f\) really has an inverse function). The theorems in this section are pretty easy to sort out; you mainly just need to remember that things like these exist.

**Chapter 7:** The focus of Chapter 7 is equivalence relations. It starts with a definition of a general relation in 7.1 and gets to equivalence relations in 7.2. You certainly need to know the definition of equivalence relation, and what reflexivity, symmetry, and transitivity are. (Note that these are all at least implicitly universally quantified; if you need to prove that a relation is symmetric, you need to consider all pairs \(a, b\) with \(aRb\) and show that \(bRa\).)

The most important nontrivial example of an equivalence relation discussed here is congruence modulo \(n\) on \(\mathbb{Z}\), but the section text and exercises give some additional examples. The importance of equivalence relations comes from the equivalence classes they define. You need to know the definition of equivalence classes and the fact that the equivalence classes of an equivalence relation form a partition of the set the relation is defined on. (So you need to know what a partition is.) Partitions and equivalence relations are different ways of describing the same thing: given a partition, we can construct a unique equivalence relation whose classes are the parts of the partition; and given an equivalence relation, we get a unique partition whose parts are the equivalence classes.

We went through some examples of equivalence relations where we can define
operations on the set of equivalence classes. The book has a whole section on
modular arithmetic and in class we constructed the rational numbers from the
integers as equivalence classes, including describing the standard operations
(and I mentioned constructing the integers from the whole numbers and the
reals from the rationals using similar approaches). The key point is that the
operations are things that take two classes and return a class, but the way we
define these operations depends on choosing elements from the classes, and so
looks like it might give different results depending on which elements we take.
So we need to check that our operations are \textit{well-defined}, meaning that we end
up with the same class no matter which elements we choose from the starting
classes.