These notes are intended to supplement Section 3.6 in the textbook. They are in preliminary form and intended only for use by students in Math 300.1 this semester. (Much of what’s here follows Dan Velleman’s book, *How to Prove It.*) I may add some material as we develop additional methods of proof.

Keep in mind that all these suggestions apply to proving a step in a larger proof as well as to a theorem or an assigned problem. So I’ll describe these in terms of trying to prove a particular goal, which could be just part of a proof of something else.

1 Proving particular kinds of statements

**To prove a goal of the form** \( P \rightarrow Q \) **directly:** Assume \( P \) is true and try to derive \( Q \) from this and known results. You can work forward from \( P \) and backward from \( Q \).

**To prove a goal of the form** \( P \rightarrow Q \) **using the contrapositive:** Assume \( Q \) is false (\( \neg Q \) is true) and try to show \( P \) is false (\( \neg P \) is true) from this and known results. You can work forward from \( \neg Q \) and backward from \( \neg P \). In general, it’s better to start with the direct approach of the preceding method, and try this one if you’re not getting anywhere with that one.

**To prove a goal of the form** \( P \rightarrow Q \) **by contradiction:** Assume the negation of \( P \rightarrow Q \), that is \( P \land \neg Q \), and try to prove a contradiction from that. The kind of contradiction you’re looking for will typically have the form \( R \land \neg R \) for some \( R \) (which will usually be something you’ve assumed as part of \( P \land Q \)).

**To prove a goal of the form** \( P \leftrightarrow Q \): Prove \( P \rightarrow Q \) and \( Q \rightarrow P \) separately.
To prove a goal of the form \( \neg P \): Try to reformulate this goal in some other form and use one of the other proof strategies in this list. Often, we can express \( \neg P \) as a conditional statement, for example. But sometimes we can’t. In that case, you can try to prove \( \neg P \) by contradiction: Assume \( P \) (\( \neg P \) is false), and try to derive a contradiction from that and known results. The kind of contradiction you’re looking for will typically have the form \( Q \land \neg Q \) for some \( Q \) (which might be the \( P \) you’re starting with).

To prove a goal of the form \( \forall x, P(x) \): Let \( x \) stand for an an arbitrary object from the appropriate universal set \( U \) and try to prove \( P(x) \). (The word arbitrary, in this kind of mathematical usage, means that \( x \) could be any element of \( U \); we’re not allowed to use anything about \( x \) other than the fact that it’s an element of the set \( U \).)

To prove a goal of the form \( \exists x, P(x) \): Try to find an element \( x \in U \) for which \( P(x) \) is true. This can be hard. Sometimes you can find one by assuming \( P(x) \) is true and working backwards to figure out what \( x \) must be. (This might involve solving an equation, or other kinds of reasoning.) But note that this working backwards won’t appear in the proof. For the proof, you’ll just start from the right \( x \) and show that \( P(x) \) is true. If this approach doesn’t work, consider a proof by contradiction, which converts the goal to \( \forall x, \neg P(x) \).

To prove a goal of the form \( P \land Q \): Consider proving \( P \) and \( Q \) separately.

To prove a goal of the form \( P \lor Q \): Assume the negation of one of the statements, and prove that the other must be true. So assume \( \neg P \) and prove \( Q \), or assume \( \neg Q \) and prove \( P \). Either of these approaches gives a proof of \( P \lor Q \).

2 Starting from a hypothesis in a particular form

To use a hypothesis of the form \( \neg P \): It can be awkward to start with a given of the form \( \neg P \). If you’re doing a proof by contradiction, you can try to prove \( \neg P \rightarrow P \) by the direct methods discussed earlier. Otherwise, you can try to reformulate \( \neg P \) in some other form.

To use a hypothesis of the form \( P \rightarrow Q \): If you also know (or can prove) that \( P \) is true, then you can conclude that \( Q \) is true and use this to continue your proof. By using the contrapositive, if you know that \( Q \) is false, you can conclude that \( P \) is false. (These inference rules are called modus ponens and modus tollens, respectively, in logic.)

For instance, if you know that \( A \rightarrow (B \rightarrow C) \) and want to prove that \( \neg C \rightarrow (A \rightarrow \neg B) \), our general approach tells us to assume \( \neg C \) and try to prove \( A \rightarrow \neg B \). So we have \( \neg C \) and \( A \rightarrow (B \rightarrow C) \) as givens, and our goal is \( A \rightarrow \neg B \). To prove \( A \rightarrow \neg B \), we assume \( A \) and try to derive \( \neg B \). So now we have \( \neg C \),
$A \rightarrow (B \rightarrow C)$, and $A$ as given. But the modus ponens rule tells us that, with these hypotheses, $B \rightarrow C$ must also be true. Since we are assuming $\neg C$, this implies $\neg B$, so we have started from $\neg C$ and the hypothesis that $A \rightarrow (B \rightarrow C)$ and we have derived $A \rightarrow \neg B$, thus proving $\neg C \rightarrow (A \rightarrow \neg B)$.

**To use a hypothesis of the form** $P \leftrightarrow Q$: Treat this as two separate givens: $P \rightarrow Q$ and $Q \rightarrow P$.

**To use a hypothesis of the form** $\exists x, P(x)$: Introduce a new variable, say $x_0$, into the proof to stand for an element of $U$ for which you know $P(x_0)$ is true. But note that you can’t assume anything about $x_0$ other than that it belongs to the universal set $U$ and $P(x_0)$ is true.

**To use a hypothesis of the form** $\forall x, P(x)$: Any time you consider an element $x_0$ of $U$ in your proof, you may use $P(x_0)$ as a true statement. Note that you can’t really make use of this immediately; you need to figure out where you want to talk about particular elements of $U$.

**To use a hypothesis of the form** $P \land Q$: Treat this as two separate givens, $P$ and $Q$.

**To use a hypothesis of the form** $P \lor Q$: Break the proof into cases, one starting with the hypothesis $P$ (and whatever other givens you have at this stage), and the other with the hypothesis $Q$ (and whatever other givens you have at this stage). Note that here the cases don’t have to be exclusive, since $P$ and $Q$ might both be true at the same time.