Exam

- Planned to be a 24-hour window starting at 8:00 am Amherst time on Tuesday, April 14 and ending at 8:00 am Amherst time on Wednesday, April 15. You will take the exam on Gradescope, submitting a separate scanned file with your answer to each problem. You’ll have 2.5 hours once you start the exam to finish it (including submitting your answers)—Gradescope enforces this. You must submit by 8:00 am EDT on April 15.

- You can use the textbook and notes from class, but no other sources (internet, books, etc.). You can’t get help from anyone except me (and you may not be able to reach me during the 2.5 hours after you’ve started the exam.)

- Exam will cover the material we’ve done since the first exam (so mostly Chapters 5, 6, and 7; I probably won’t ask you anything from Section 4.3). You won’t have done homework on Chapter 7, but it will be on the exam.

- We’ll do review on Thursday in class, I will hold Zoom office hours Thursday and Monday at 11:00.

- I’ll post a review sheet later this week.

- We won’t have class on Tuesday, April 14.
Constructing $\mathbb{Q}$

In the previous class, we constructed $\mathbb{Z}_n$ as a set of equivalence classes for an equivalence relation (congruence mod $n$) on $\mathbb{Z}$. We defined addition and multiplication operations on $\mathbb{Z}_n$ (which is a little tricky because the definitions work by using regular integers that are elements of the classes, meaning we have to show the operations are well-defined). In this class, we’ll construct $\mathbb{Q}$ as a set of equivalence classes with appropriate operations, etc.

What’s the goal?

- In $\mathbb{Z}$, we can’t solve most equations of the form $bx = a$, because there’s no element that gives $a$ when multiplied by $b$. (There’s no quotient $a \div b$.)
- Want to build a bigger set (we’re extending $\mathbb{Z}$) in which we can solve all possible $bx = a$ (i.e., not for all $a$ when $b = 0$). So we need to add in all these quotients.
- Want this to have operations that extend the operations on $\mathbb{Z}$.
How to “construct” quotients

- Given $a, b$, with $b \neq 0$, we need an element that will become the quotient $a \div b$. So we start with pairs $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ with $b \neq 0$.

- But there are many pairs that give the same quotient: $(1, 2)$, $(2, 4)$ $(3, 6)$, etc. So we need to collapse these to get a single “rational number” that will be the quotient.

- How do we do this?
  - If $bx = a$ and $dx = c$, then $ac = bcx = adx$. So, if $x \neq 0$, we should have $ad = bc$. Conversely, if $ad = bc$ and we can divide, we’ll get $a \div b = c \div d$.
  - So two pairs $(a, b)$ and $(c, d)$ should get the same quotient if and only if $ad = bc$. 

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc$$
The set and equivalence relation

So we let \( X = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | b \neq 0\} \) and we take \( \sim \) to be the relation on \( X \) defined by

\[ (a, b) \sim (c, d) \iff ad = bc. \]

The set \( X \) consists of pairs that describe a quotient, and two pairs are related by \( \sim \) if they represent the same quotient.

We have to check that \( \sim \) is an equivalence relation:

- Is \( \sim \) is reflexive? \( (a, b) \sim (a, b) \)? \( ab = ab \) \( \checkmark \)

- Is \( \sim \) is symmetric? \( (a, b) \sim (c, d) \rightarrow (c, d) \sim (a, b) \)? \( ab = bc \rightarrow cb = da \) \( \checkmark \)

- Is \( \sim \) transitive? \( (a, b) \sim (c, d) \land (c, d) \sim (e, f) \rightarrow (a, b) \sim (e, f) \)?

\[ \begin{align*}
  ab &= bc \quad \text{(c.o.)} \\
  cf &= df \quad \text{(c.o.)} \\
  a \cdot d \cdot c \cdot f &= b \cdot c \cdot d \cdot e \\
  \frac{a\cdot d\cdot c\cdot f}{c} &= \frac{b\cdot c\cdot d\cdot e}{c} \\
  \frac{ad}{d} &= \frac{be}{e} \\
  \frac{a}{d} &= \frac{b}{e} \\
  a \cdot d \neq c \\
  c \neq 0 \rightarrow a = 0 \land c = 0 \implies a \sim c \\
  \begin{align*}
    \frac{a \cdot d \cdot f}{d} &= \frac{b \cdot d \cdot e}{d} \\
    \frac{a}{d} &= \frac{b}{e} \end{align*}
\]
Operations on the set of equivalence classes

So let $\tilde{X} = X / \sim$ be the set of equivalence classes for the relation $\sim$ on $X$. Note that $[(a, b)] = [(na, nb)]$ for all $n \neq 0$ in $\mathbb{Z}$. We now define operations on $\tilde{X}$:

Let

$[(a, b)] \oplus [(c, d)] = [(ad + bc, bd)]$

This really means: For $\bar{x}, \bar{y} \in \tilde{X}$, define $\bar{x} \oplus \bar{y}$ as follows:

1. Choose a pair $(a, b) \in \bar{x}$. \underline{representative for $\bar{x}$}
2. Choose a pair $(c, d) \in \bar{y}$. 
3. Form the pair $(ad + bc, bd)$ from these. \underline{representatives}
4. Take $\bar{x} \oplus \bar{y}$ to be $[(ad + bc, bd)]$.

$x/\sim$ set of equiv classes
$f(x)$ modulo $\sim$

$[(1, 2)] = [(39, 78)]$

Each class has infinitely many pairs

Prove $\oplus$ is well-defined.
Since our definition of $\bar{x} \oplus \bar{y}$ depends on choosing elements from the two equivalence classes, we need to make sure that the resulting equivalence class doesn’t depend on those choices, but just on the two equivalence classes $\bar{x}$ and $\bar{y}$.

To check this, suppose you choose $(a, b)$ from $\bar{x}$ and $(c, d)$ from $\bar{y}$, while I choose $(a', b')$ from $\bar{x}$ and $(c', d')$ from $\bar{y}$. So $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. We have to show that $(ad + bc, bd) \sim (a'd' + b'c', b'd')$ so that we get the same class for $\bar{x} \oplus \bar{y}$.

Since $ab' = a'b$ and $cd' = c'd$, we have

\[
\begin{align*}
(a, b) \sim (a', b') & \quad \text{and} \quad (c, d) \sim (c', d') \\
(ad + bc) b' d' & = a db' d' + c b b' d' = a'd' b d' + c' b b' d' \\
& = (a'd' + b'c') b d
\end{align*}
\]

So $(ad + bd, bd) \sim (a'd' + b'c', b'd')$. This means that $\oplus$ is well-defined.
Properties of \( \oplus \)

- \( \oplus \) is commutative.
  \[
  x \oplus y = y \oplus x
  \]
  \[
  [(a, b) + (c, d), (e, f)] = [(c, d) + (a, b), (e, f)]
  \]

- \( \oplus \) is associative.
  \[
  [(a, b) + (c, d), (e, f)] = [(c, d) + (a, b), (e, f)]
  \]

- \([0, 1]\) is an identity element for \( \oplus \).
  \[
  [(0, 1), (a, b)] = [(a, b), (0, 1)] = [(a, b)]
  \]

- For all \([a, b]\), \([(a, b)] \oplus [(-a, b)] = [(0, 1)]
  \[
  [(a, b)] \oplus [(-a, b)] = [(a, b) + (-a, b), (0, 0)]
  \]
  \[
  = [(0, 0)]
  \]
  \[
  = [(0, 1)]
  \]

So our addition is commutative and associative and we have additive inverses and subtraction.

Defined an operation on \( \overline{X} \)

"like" + on \( \mathbb{Z} \)
We compute $\tilde{x} \otimes \tilde{y}$ as follows:

1. Choose a pair $(a, b) \in \tilde{x}$.
2. Choose a pair $(c, d) \in \tilde{y}$.
3. Form the pair $(ac, bd)$ in $\mathbb{Z} \times \mathbb{Z}$.
4. Let $\tilde{x} \otimes \tilde{y} = [(ac, bd)]$.

Since this calculation also requires the choice of elements from $\tilde{x}$ and $\tilde{y}$, we have to show that it is well-defined, too.
is well-defined

As for $\oplus$, take $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. We need to show that 

$$(ac, bd) \sim (a'c', b'd') \quad \text{or} \quad acb'd' = a'c'bd.$$ 

$$acb'd' = a'cbd' \quad \text{since} \quad ab' = a'b$$

$$= a'c'bd \quad \text{since} \quad cd' = c'd$$

So $\otimes$ is well-defined. We can check that:

- $\otimes$ is commutative and associative
- $[(1, 1)]$ is an identity element for $\otimes$
- If $[(a, b)] \neq [(0, 1)]$, then $a \neq 0$ and $[(a, b)] \otimes [(b, a)] = [(1, 1)]$. (So we have a multiplicative inverse for each nonzero element.)
- The distributive laws hold.

\[
\begin{align*}
[(3, 1)] \otimes [(a, b)] &= [(3 \cdot a, 1 \cdot b)] \\
&= [(a, b)]
\end{align*}
\]

\[
\begin{align*}
[(a, b)] [(b, a)] &= [(ab, ab)] \\
&= [(1, 1)]
\end{align*}
\]
How does $\mathbb{Z}$ fit into this?

So we now have a set (of equivalence classes) with addition and multiplication operations, inverses, etc. But how does this solve our original problem of adding elements to $\mathbb{Z}$?

The function $i: \mathbb{Z} \rightarrow \tilde{X}$ defined by $i(z) = [(z,1)]$ is injective. So we can identify each $z \in \mathbb{Z}$ with its image $i(z)$ in $\tilde{X}$.

And the function $i$ preserves the operations on $\mathbb{Z}$:

- $i(z_1 + z_2) = i(z_1) \oplus i(z_2)$
- $i(z_1 z_2) = i(z_1) \otimes i(z_2)$

So we think of $\mathbb{Z}$ as a subset of $\tilde{X}$ (the subset is $i(\mathbb{Z})$), with its regular operations matching up with $\oplus$ and $\otimes$ on that subset.

$$i(z_1 + z_2) = i(z_1) \oplus i(z_2) \quad \text{add in } \mathbb{Z} \quad \text{addition in } \tilde{X}$$

$$i(z_1) \oplus i(z_2) = [(z_1,1)] \oplus [(z_2,1)] = [(z_1 + z_2,1)]$$

Every integer corresponds to an element of $\tilde{X}$.

$z \mapsto i(z) = [(z,1)]$ corresponds
Now we clean up the notation:

We call $\bar{X} \otimes \mathbb{Q}$ instead, and write $+ \oplus$ for $\oplus$ and $\times \otimes$ for $\otimes$. And we think of $\mathbb{Z}$ as being the classes $[(z, 1)]$ in $\mathbb{Q}$.

One more piece of notation. We write $\frac{a}{b}$ instead of $[(a, b)]$. But note that this notation specifies not just an equivalence class, but a particular element, $(a, b)$ of the equivalence class. In this sense, the “fractions” $\frac{1}{2}$ and $\frac{2}{4}$ are the same rational number but different (though equivalent) representatives for that number.

So “equivalent fractions” are equal as rational numbers but unequal as representations of that rational number. And when we calculate with rational numbers, it’s very often convenient to change to a different representation, replacing fractions with equivalent ones.

\[
\frac{1}{2} = \frac{3}{6} \quad \text{as rational numbers} \quad (1, 2) \sim (3, 6)
\]

\[
[(1, 2)] = [(3, 6)] \quad \text{but} \quad (1, 2) \neq (3, 6)
\]
Constructing $\mathbb{Z}$ from $\mathbb{W}$

We can use the same sort of process to construct $\mathbb{Z}$ from $\mathbb{W}$. Here, we’re trying to make sure we have all the differences $a - b$. So we take pairs representing differences, define a relation to make two pairs equivalent if they represent the same difference (e.g., (2, 3) and (11, 12)), and define operations for addition and multiplication on the set of equivalence classes.

We show that there’s an injection from $\mathbb{W}$ to the set of equivalence classes that matches up the operations in $\mathbb{W}$ with the ones on our set of equivalence classes, and we say that the set of equivalence classes is $\mathbb{Z}$, with $\mathbb{W}$ identified with its image in $\mathbb{Z}$.
to construct \( \mathbb{R} \) from \( \mathbb{Q} \) need some "geometry"

sequences of rationals that don't converge even though they should
no limit in \( \mathbb{Q} \)

\[
1, 1.4, 1.41, 1.414, \ldots \rightarrow \sqrt{2}, \text{ but this isn't in } \mathbb{Q}
\]
so in \( \mathbb{Q} \), this sequence doesn't converge

A sequence \( \{a_i\} \) is Cauchy if for all \( \varepsilon > 0 \), there is an \( N \in \mathbb{N} \)
such that \( i, j \geq N \rightarrow |a_i - a_j| < \varepsilon \)

for Cauchy, terms of sequence are all within \( \varepsilon \) of each other.

in \( \mathbb{R} \), Cauchy \( \leftrightarrow \) convergent. But not true in \( \mathbb{Q} \)
But can't take \( \mathbb{Q} \) to be Cauchy sequences of rationals because different sequences can have same limit.

Take an equivalence relation that expresses the sequences having same limit (get really close to each other)

Take set of equivalence classes,

define operations (and show well-defined)

show how to identify \( \mathbb{Q}, +, \cdot \) with a subset of set of classes and its operations.