Math 235H—Spring 2017
Solutions for Homework due Thursday, April 27

Section 4.4

3. (a) If $A$ has three orthogonal columns each of length 4, what is $A^T A$?

If the columns are $a_1, a_2, a_3$, we know that $a_i^T a_j$ is 16 when $i = j$ and 0 when $i \neq j$. The $(i,j)$ entry in $A^T A$ is $a_i^T a_j$, so $A^T A = 16I$.

(b) If $A$ has three orthogonal columns of lengths 1, 2, 3, what is $A^T A$?

If the columns $a_1, a_2, a_3$ have lengths 1, 2, 3, respectively, then $a_i^T a_i = i^2$ and $a_i^T a_j = 0$ for $i \neq j$. So the $A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$.

• If $q_1$ and $q_2$ are orthonormal vectors in $\mathbb{R}^5$, what combination $\underline{q}_1 + \underline{q}_2$ is closest to a given vector $b$?

The orthogonal projection of $b$ onto the subspace spanned by $q_1$ and $q_2$ is the linear combination of $q_1$ and $q_2$ that is closest to $b$. We know that the projection matrix $P = Q(Q^T Q)^{-1}Q^T$, where $Q$ is the matrix with the $q_i$ as its columns, maps $b$ to this projection. Since the $q_i$ are orthonormal, $Q^T Q = I_2$, and $P = QQ^T$. Thus, the orthogonal projection of $b$ is $QQ^T b = (q_1^T b)q_1 + (q_2^T b)q_2$.

18. Find orthogonal vectors $A, B, C$ by Gram-Schmidt from $a = (1, -1, 0, 0)$, $b = (0, 1, -1, 0)$, $c = (0, 0, 1, -1)$.

We find orthogonal vectors $A, B, C$ from Gram-Schmidt, but the problem doesn’t ask us to normalize the lengths to 1. We take $A = a$. We then set $B = b - \frac{a^T A}{A^T A} A = (\frac{1}{2}, \frac{1}{2}, -1, 0)$. Finally, we take $C = c - \frac{A^T A}{A^T A} A - \frac{B^T B}{B^T B} B = (0, 0, 1, -1) - 0 + \frac{1}{2}(\frac{1}{2}, \frac{1}{2}, -1, 0) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

Section 5.1

1. If a 4 by 4 matrix has det $A = \frac{1}{2}$, find det$(2A)$ and det$(-A)$ and det$(A^2)$ and det$(A^{-1})$.

If we multiply a $4 \times 4$ matrix by a scalar $a$, we multiply the determinant by $a^4$. So det$(2A) = 16(\frac{1}{2}) = 8$ and det$(-A) = \det(A) = \frac{1}{2}$. For two $n \times n$ matrices $A, B$, we know that det$(AB) = (\det A)(\det B)$, so det$(A^2) = (\det A)^2 = \frac{1}{4}$. And det$(A^{-1}) = \frac{1}{\det A} = 2$. 

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4. Which row exchanges show that these “reverse identity matrices” $J_3$ and $J_4$ have $|J_3| = -1$ but $|J_4| = 1$?

$$\begin{vmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{vmatrix} = -1 \quad \text{but} \quad \begin{vmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{vmatrix} = 1.$$

We obtain $J_3$ by exchanging the first and third rows of $I_3$. This is one row exchange, so it multiplies the determinant by $-1$. We obtain $J_4$ by, for example, exchanging the first and fourth rows of $I_4$ and then exchanging the second and third rows. Doing these two row exchanges multiplies the determinant by $(-1)^2$.

7. Find the determinants of rotations and reflections:

$$Q = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix}
1 - 2 \cos^2 \theta & -2 \cos \theta \sin \theta \\
-2 \cos \theta \sin \theta & 1 - 2 \sin^2 \theta
\end{bmatrix}.$$

For the first matrix, which is rotation counterclockwise through $\theta$, the determinant is $\cos^2 \theta + \sin^2 \theta = 1$. For the second matrix, which is reflection through the line perpendicular to $(\cos \theta, \sin \theta)$, the determinant is $(1 - 2 \cos^2 \theta)(1 - 2 \sin^2 \theta) - 4 \cos^2 \theta \sin^2 \theta = 1 - 2(\cos^2 \theta + \sin^2 \theta) = -1$.

8. Prove that every orthogonal matrix ($Q^T Q = I$) has determinant 1 or $-1$.

(a) Use the product rule $|AB| = |A||B|$ and the transpose rule $|Q| = |Q^T|$.

We know that $\det(QQ^T) = \det(Q)\det(Q^T)$ and that $\det(Q^T) = \det(Q)$. So $\det(QQ^T) = (\det(Q))^2 = \det(I) = 1$. It follows that $\det(Q) = \pm 1$.

(b) Use only the product rule. If $|\det(Q)| > 1$, then $\det Q^n = (\det Q)^n$ blows up. How do you know this can’t happen to $Q^n$?

Observe that if $Q$ is orthogonal, so is $Q^n$. This is because $(Q^n)^T = (Q^T)^n$ (since we know $(AB)^T = B^T A^T)$ so $(Q^n)^T = QQ\ldots QQ^T Q^T \ldots Q^T$, where each of $Q$ and $Q^T$ occurs $n$ times. But each $Q$ gets multiplied by a $Q^T$ to produce the identity matrix.

At this point, Strang expects you to say that the determinant of an orthogonal matrix can’t be arbitrarily large (i.e., that it’s not the case that, given any $N > 0$, you can find an orthogonal matrix whose determinant is greater than $N$). But I’m not sure you actually know enough (without reading Section 5.2 or 5.3, anyway) to do this. Here are a
couple of ways you could do it. One is to say that the determinant of a matrix is the volume of the “box” formed by the columns. Since $Q^n$ is orthogonal, the length of each column is 1, and so the biggest the volume can be is when they form a rectangular parallelepiped ("parallelepiped" is the technical term for the $n$-dimensional box formed by the vectors when $n > 3$), i.e., when the vectors are all perpendicular to each other. That (signed) volume has absolute value 1. But if you do this you don’t need to think about $Q^n$, just $Q$, so it’s not what Strang is aiming you at. Another way is to observe (really from the “big formula”) that the determinant is a continuous function of the entries of the matrix (because it’s a sum of products of the entries) and that the entries must all have absolute value $\leq 1$ since the columns (and rows) have length 1. So we’re evaluating a continuous function on a closed region, and it must take a maximum value at some point in the region. If you take $N$ larger than that maximum value, you see that the determinant of an orthogonal matrix can’t be greater than $N$, so the determinant of $Q^n$ can’t get arbitrarily large as $n$ goes to infinity. (And since the determinant of the orthogonal matrix $Q^{-1} = Q^T$ is $\frac{1}{\det Q}$, if the determinant of $Q$ has absolute value less than 1, the determinant of $Q^T$ has absolute value greater than 1. So we’ve taken care of that case, too.) I think this second argument is what Strang really has in mind, but I don’t think it’s really plausible to expect you to come up with this in Section 5.1. Of course, you all have read Sections 5.2 and 5.3 by now, so...

10. If the entries in every row of $A$ add to zero, solve $Ax = 0$ to prove $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean $\det A = 1$?

If the sum of entries in each row is 0, we know that $A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0$. But then there is a nonzero vector in the nullspace of $A$. So $A$ is singular and $\det A = 0$.

If every row adds to 1, then $A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, so $(A - I) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0$

and the nullspace of $A - I$ contains a nonzero vector. Then $\det(A - I) = 0$. But this does not mean that $\det A = 1$. Con-
sider, for example, the matrix \[
\begin{bmatrix}
1 & 0 \\
\frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\], whose rows add to 1 but whose determinant is \(\frac{1}{2}\).

28. True or false (give a reason if true or a 2 by 2 counterexample if false):

(a) If \(A\) is not invertible, then \(AB\) is not invertible.

True. If \(\det A = 0\), then \(\det(AB) = (\det A)(\det B) = 0\), so \(AB\) is not invertible. (Note that this only makes sense if \(A\) and \(B\) are square matrices of the same size.)

(b) The determinant of \(A\) is always the product of its pivots.

False. If an odd number of row exchanges are necessary for elimination, the determinant of \(A\) will be \(-1\) times the product of its pivots. Consider \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\].

(c) The determinant of \(A - B\) equals \(\det A - \det B\).

False. Consider \(A = 3I_2\) and \(B = 2I_2\). Then \(A - B = I_2\) has determinant 1 but \(\det A = 9\) and \(\det B = 4\).

(d) \(AB\) and \(BA\) have the same determinant.

True. Since \(\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA)\).

Section 5.2

1A. Compute the determinant of \(A = \begin{bmatrix}
1 & 2 & 3 \\
3 & 1 & 2 \\
3 & 2 & 1
\end{bmatrix}\) from six terms [the "big formula"]. Are the rows independent?

The determinant of \(A\) is \(a_{1,1}a_{2,2}a_{3,3} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,3}a_{3,1} = 1 - 4 - 6 + 12 + 18 - 9 = 12\). Since the determinant is nonzero, the matrix is nonsingular and its rows must be independent.

3. Show that \(\det A = 0\), regardless of the five nonzeros marked by \(x\)’s:

\[
A = \begin{bmatrix}
x & x & x \\
0 & 0 & x \\
0 & 0 & x
\end{bmatrix}
\]

What are the cofactors of row 1? What is the rank of \(A\)? What are the 6 terms in \(\det A\)?

If we expand the determinant by cofactors along the first row, we see that each of the submatrices (obtained by deleting the first row and one of the columns) has a column of 0s and
therefore has determinant 0. So each cofactor of row 1 is 0 and the determinant of \( A \) is 0, regardless of the values of the \( x \) entries.

The rank of \( A \) is 2 (assuming all the \( x \) entries are nonzero) since the third row is a nonzero multiple of the second. So \( A \) must be singular and therefore have determinant 0.

Each of the six terms must include an entry from each row and column. But all the nonzero entries lie in the first row and third column, so each of the six terms must have a factor of 0. This also shows that the determinant is 0.

5. Place the smallest number of zeros in a 4 by 4 matrix that will guarantee \( \det A = 0 \). Place as many zeros as possible while still allowing \( \det A \neq 0 \).

Four zeros in a single row or a single column will guarantee that the determinant is 0. With only three zeros, it’s not possible to guarantee that the determinant will be 0 (no matter what the nonzero entries are) since there will always be a nonzero elementary product no matter where the three zeros are placed.

The identity matrix \( I_4 \) has 12 zeros out of its 16 entries. It’s not possible to have 13 zeros in a 4 by 4 matrix and a nonzero determinant, since every elementary product would have to involve a 0.

11. Find all cofactors and put them into cofactor matrices \( C, D \). Find \( AC \) and \( \det B \).

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}.
\]

The cofactor matrix for \( A \) is \( C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \). \( AC = \begin{bmatrix} ad - b^2 & ba - ac \\ cd - bd & da - c^2 \end{bmatrix} \).

(Note that \( AC^T = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (\det A)I \).)

The cofactor matrix of \( B \) is \( \begin{bmatrix} 0 & -42 & 35 \\ -3 & 6 & -3 \end{bmatrix} \). Using the cofactor expansion along the third row, we have \( \det B = 7(-3) + 0 + 0 = -21 \).

12. Find the cofactor matrix \( C \) and multiply \( A \) times \( C^T \). Compare \( AC^T \) with \( A^{-1} \).

\[
A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.
\]
The cofactor matrix of $A$ is $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. So $AC^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. So $AC^T = 4I$ and $A^{-1} = \frac{1}{4}C^T$. 