Section 3.4

7. If \( \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \) are independent vectors, show that the differences \( \mathbf{v}_1 = \mathbf{w}_2 - \mathbf{w}_1, \mathbf{v}_2 = \mathbf{w}_1 - \mathbf{w}_3, \) and \( \mathbf{v}_3 = \mathbf{w}_1 - \mathbf{w}_2 \) are dependent. Find a combination of the \( \mathbf{v}s \) that gives zero. What matrix \( A \) in \( \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{bmatrix} \) is singular?

It’s easy to check that \( \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0} \). So this is a nontrivial (not all coefficients 0) linear combination of the differences that is the zero vector, showing that the \( \mathbf{v}_i \) are dependent. The matrix whose columns are the \( \mathbf{v}_i \) is therefore singular, since its nullspace has dimension at least 1.

16. Find a basis for each of these subspaces of \( \mathbb{R}^4 \):

(a) All vectors whose component are equal.

One basis is \( \{(1,1,1,1)\} \). (There are infinitely many bases, of course, since the set whose only element is any nonzero multiple of \((1,1,1,1)\) will also be a basis.)

(b) All vectors whose components add to zero.

This subspace is the nullspace of the matrix \( A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \), i.e., the set of vectors with \( x_1 + x_2 + x_3 + x_4 = 0 \). This is in reduced row echelon form and we can read off a basis for the nullspace, getting \((−1,1,0,0),(−1,0,1,0),(−1,0,0,1)\) (Again, there are infinitely many bases for this subspace.)

(c) All vectors that are perpendicular to \((1,1,0,0)\) and \((1,0,1,1)\).

This is the nullspace of the matrix \( A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \) whose rows are the two given vectors. Subtracting the first row from the second and then adding the second to the first gives the matrix \( R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \) in reduced row echelon form. A basis for the nullspace is obviously \( \{(-1,1,1,0),(-1,1,0,1)\} \).

(d) The column space and the nullspace of \( I \ (4 \times 4) \).

The column space is clearly all of \( \mathbb{R}^4 \), so we can take any basis for \( \mathbb{R}^4 \). But the column vectors in the identity matrix are independent (they form what’s called the standard basis), so we might as well use them. The nullspace is \( Z \), consisting only of the zero vector. The vector \( \mathbf{0} \) by itself can’t be a basis because \( 1 \cdot \mathbf{0} = \mathbf{0} \) is a nontrivial linear combination equal to the zero vector. The usual convention adopted for this is to say that the empty set is a basis for \( Z \).
34. Suppose $y_1(x), y_2(x), y_3(x)$ are three different functions of $x$. The vector space they span could have dimension 1, 2, or 3. Give an example of $y_1, y_2, y_3$ to show each possibility.

To get a subspace of dimension 1, the functions must all be scalar multiples of one of them. One obvious choice is $x, 2x, 3x$. But something like $0, x, 2x$ would work, too.

To get a subspace of dimension 2, one of the vectors must be a linear combination of the other two, but those other two must be independent. So we could take $x, x+1, 2x+1$ or $x, 2x, x^2$. For dimension 3, the functions must be independent. One example is $1, x, x^2$ (where 1 is the constant function whose value is 1).

Note that there are infinitely many correct answers to this problem.

Section 4.1

13. Put bases for the subspaces $V$ and $W$ into the columns of matrices $V$ and $W$. Explain why the test for orthogonal subspaces can be written $V^T W = 0$ matrix. This matches $v^T w$ for vectors.

Saying $V^T W = 0$ says that each vector in the basis for $V$ is orthogonal to each vector in the basis for $W$. If $\{v_1, \ldots, v_n\}$ is the basis for $V$ and $\{w_1, \ldots, w_m\}$ is the basis for $W$, every vector in $V$ can be written in the form $\sum_{i=1}^n a_i v_i$ for some real numbers $a_1, \ldots, a_n$ and every vector in $W$ can be written in the form $\sum_{j=1}^m b_j w_j$ for some real numbers $b_1, \ldots, b_m$. But the dot product $(a_1 v_1 + \cdots + a_n v_n) \cdot (b_1 w_1 + \cdots + b_m w_m)$ is just a sum of multiples of the $v_i \cdot w_j$, by the distributive property of the dot product. And $V^T W = 0$ says all the $v_i \cdot w_j$ are 0. So every vector in $V$ is orthogonal to every vector in $W$.

15. Extend Problem 14 to a $p$-dimensional subspace $V$ and a $q$-dimensional subspace $W$ of $\mathbb{R}^n$. What inequality on $p+q$ guarantees that $V$ intersects $W$ in a nonzero vector? These subspaces cannot be orthogonal.

Let $\{v_1, \ldots, v_p\}$ be a basis for $V$ and $\{w_1, \ldots, w_q\}$ be a basis for $W$. If $p+q > n$, then the set $\{v_1, \ldots, v_p, w_1, \ldots, w_q\}$ can’t be independent, because it has more vectors than the dimension of $\mathbb{R}^n$. So there’s some nontrivial (not all coefficients are 0) linear combination $a_1 v_1 + \cdots + a_p v_p + b_1 w_1 + \cdots + b_q w_q = 0$. Separating that out, we have $a_1 v_1 + \cdots + a_p v_p = -b_1 w_1 - \cdots - b_q w_q$, where the two sums on opposite sides of the equal sign are both nonzero. (Why both?) But the vector on the left side is in $V$ and the vector on the right is in $W$, so there’s some vector in both subspaces and their intersection isn’t just the zero vector. And, since that nonzero vector in the intersection isn’t orthogonal to itself, the two subspaces can’t be orthogonal.
17. If \( S \) is the subspace of \( \mathbb{R}^3 \) containing only the zero vector, what is \( S^\perp \)?

Every vector is orthogonal to the zero vector, so \( S^\perp \) is all of \( \mathbb{R}^3 \).

If \( S \) is spanned by \((1,1,1)\), what is \( S^\perp \)?

Since \( S \) consists of all the scalar multiples of \((1,1,1)\), the vectors orthogonal to all of \( S \) are just the vectors orthogonal to \((1,1,1)\). These are the solutions to \( x + y + z = 0 \), or the nullspace of the matrix \[
\begin{bmatrix}
1 & 1 & 1
\end{bmatrix}
\]. It’s easy to see that a basis for this is \(\{(1,-1,0),(1,0,-1)\}\).

If \( S \) is spanned by \((1,1,1)\) and \((1,1,-1)\), what is a basis for \( S^\perp \)?

The space \( S^\perp \) is the nullspace of the matrix \[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & -1
\end{bmatrix}
\]. Subtracting the first row from the second gives the matrix \[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & -2
\end{bmatrix}
\]. Dividing the second row by \(-2\) and then subtracting the second row from the first gives the matrix \(R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) in reduced row echelon form. So a basis for the nullspace is \(\{(1,-1,0)\}\).

23. If a subspace \( S \) is contained in a subspace \( V \), prove that \( S^\perp \) contains \( V^\perp \).

We need to show that every vector that is orthogonal to all of \( V \) is orthogonal to all of \( S \).

But suppose \( s \) is some vector in \( S \) and \( v' \) is some vector in \( V^\perp \). So \((v'^T)v = 0\) for every \( v \) in \( V \). Since \( s \in S \) and \( S \) is contained in \( V \), \( s \in V \). Thus, \( v' \) is orthogonal to \( s \) for every \( s \in S \).

30. Suppose \( A \) is 3 by 4 and \( B \) is 4 by 5 and \( AB = 0 \). So \( N(A) \) contains \( C(B) \).

Prove from the dimensions of \( N(A) \) and \( C(B) \) that \( \text{rank}(A)+\text{rank}(B) \leq 4 \).

We know that the dimension of \( N(A) \) is \( 4 - \text{rank}(A) \), and that \( \text{rank} B \) is the dimension of \( C(B) \), which must be less than or equal to \( N(A) \), since \( C(B) \) is contained in \( N(A) \). So \( 4 - \text{rank}(A) \geq \text{rank}(B) \), which is the same as saying \( 4 \geq \text{rank}(A) + \text{rank}(B) \).

Section 4.2

1a. Project the vector \( b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \) onto the line through \( a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \).

We saw that projection of \( b \) onto the line through \( a \) is \( p = \frac{a^T b}{a^T a} a \). Here, \( p = \frac{2}{3}(1,1,1) \), so \( e = (-\frac{2}{3},\frac{2}{3},\frac{2}{3}) \).
6. Project \( \mathbf{b} = (1, 0, 0) \) onto the lines through \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \) in Problem 5 and also onto \( \mathbf{a}_3 = (2, -1, 2) \). Add up the three projections \( p_1 + p_2 + p_3 \).

The projection of \( \mathbf{b} \) onto the line through \( \mathbf{a}_1 \) is \( \frac{\mathbf{a}_1 \cdot \mathbf{b}}{\mathbf{a}_1 \cdot \mathbf{a}_1} \mathbf{a}_1 = -\frac{1}{9}(-1, 2, 2) \). Similarly, the projection of \( \mathbf{b} \) onto the line through \( \mathbf{a}_2 \) is \( \frac{2}{5}(2, 2, -1) \) and the projection of \( \mathbf{b} \) onto the line through \( \mathbf{a}_3 \) is \( \frac{2}{5}(2, -1, 2) \). Adding these up gives \( \mathbf{b} \).

11. Project \( \mathbf{b} \) onto the column space of \( \mathbf{A} \) by solving \( \mathbf{A}^T \mathbf{A} \hat{\mathbf{x}} = \mathbf{A}^T \mathbf{b} \) and \( \mathbf{p} = \mathbf{A} \hat{\mathbf{x}} \). Find \( \mathbf{e} = \mathbf{b} - \mathbf{p} \). It should be perpendicular to the columns of \( \mathbf{A} \).

(a) \( \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \).

\( \mathbf{A}^T \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \) and \( \mathbf{A}^T \mathbf{b} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \), so \( \hat{\mathbf{x}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \). Then \( \mathbf{p} = \mathbf{A} \hat{\mathbf{x}} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \), and \( \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \), which is indeed perpendicular to the columns of \( \mathbf{A} \).

(b) \( \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} \).

In the same way, we get \( \hat{\mathbf{x}} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \), so \( \mathbf{p} = \mathbf{A} \hat{\mathbf{x}} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \mathbf{b} \) and \( \mathbf{e} = \mathbf{0} \). In this case, \( \mathbf{b} \) is actually in the column space of \( \mathbf{A} \), so its projection, the vector in the column space of \( \mathbf{A} \) that’s closest to \( \mathbf{b} \), is just itself.

25. The projection matrix \( \mathbf{P} \) onto an \( n \)-dimensional subspace of \( \mathbb{R}^m \) has rank \( r = n \). **Reason:** The projections \( \mathbf{P} \mathbf{b} \) fill the subspace \( \mathbf{S} \). So \( \mathbf{S} \) is the column space of \( \mathbf{P} \).

The column space of the projection matrix is the set of projections of vectors in \( \mathbb{R}^m \) onto the subspace. Since projecting a vector in the subspace onto the subspace just gives that vector back, we see that the column space must be all the vectors in the subspace. So the dimension of the column space of the projection matrix \( \mathbf{P} \) is the dimension of the subspace \( \mathbf{P} \) projects onto. \( \mathbf{S} \) is the column space of \( \mathbf{P} \).

31. In \( \mathbb{R}^m \), suppose I give you \( \mathbf{b} \) and also a combination \( \mathbf{p} \) of \( \mathbf{a}_1, \ldots, \mathbf{a}_n \). How would you test to see if \( \mathbf{p} \) is the projection of \( \mathbf{b} \) onto the subspace spanned by the \( \mathbf{a}_i \)?
If $p$ is the projection of $b$ onto the space spanned by the $a_i$, then $e = b - p$ must be orthogonal to the space spanned by the $a_i$. So it’s enough to check that each $a_i$ is perpendicular to $b - p$.

**Section 4.3**

1. With $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$ set up and solve the normal equations $A^T A \hat{x} = Ab$. For the best straight line in Figure 4.9a, find its four heights $p_i$ and four errors $e_i$. What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

For this problem, we are trying to find the line $f(t) = a_0 + a_1 t$ that best fits the data. So our four equations are $1a_0 + 0a_1 = 0$, $1a_0 + 3a_1 = 8$, etc. That makes $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. We have $A^T A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^T b = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$. Solving $A^T A \hat{x} = A^T b$ give $\hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, so the line we want is $f(t) = 1 + 4t$. The vector $p$ giving the values on our line for $t = 0, 1, 3, 4$ is $p = A \hat{x} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and the $e_i$, measuring how much the values on our line differ from the actual data collected, are the components of $e = b - e = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$, so $E = \|e\|^2 = 44$.

5. Find the height $C$ of the best horizontal line to fit $b = (0, 8, 8, 20)$. An exact fit would solve the unsolvable equations $C = 0, C = 8, C = 8, C = 20$. Find the 4 by 1 matrix $A$ in these equations and solve $A^T A \hat{x} = A^T b$.

Draw the horizontal line at height $\hat{x} = C$ and the four errors in $e$.

In this case, we want to find the function $f(t) = C$ that best fits the collected data. The problem gives the equations; the matrix $A$ of coefficients for those equations is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. $A^T A = [4], (A^T A)^{-1} = [\frac{1}{4}]$ and $A^T b = [36]$. So $[C] = A(A^T A)^{-1} A^T b = [9]$. The horizontal line at height 9 is the best horizontal approximation. In this case, $e = b - A \hat{x} =$
(-9, -1, -1, 11) is easy to compute by subtracting 9 from each of the data values, and the errors add to 0.

9. For the closest parabola \( b = C + Dt + Et^2 \) to the same four points, write down the unsolvable equations \( Ax = b \) in three unknowns \( x = (C, D, E) \). Set up the three normal equations \( A^T A \hat{x} = A^T b \) (solution is not required). In Figure 4.9a, you are now fitting a parabola to 4 points—what is happening in Figure 4.9b?

The equation for \( t = 0 \), for example, is \( 1C + 0D + 0^2E + = 0 \) and the one for \( t = 3 \) is \( 1C + 3D + 3^2E = 8 \). So the coefficient matrix for the unsolvable equations is \( \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \) and we have the unsolvable system of equations

\[
\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}.
\]

Then \( A^T A = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \). So the normal equations are given by

\[
\begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \end{bmatrix}.
\]

10. For the closest cubic \( b = C + Dt + Et^2 + Ft^3 \) to the same four points, write down the four equations \( Ax = b \). Solve them by elimination. In Figure 4.9a, this cubic now goes exactly through the points. What are \( p \) and \( e \)?

The equations are

\[
\begin{align*}
C + 0D + 0E + 0F &= 0 \\
C + D + E + F &= 8 \\
C + 3D + 9E + 27F &= 8 \\
C + 4D + 16E + 64F &= 20
\end{align*}
\]

In matrix form, this is

\[
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}.
\]

In this case the system is consistent (since we can always find a cubic
that goes through any 4 given points in the plane; in general, we

\[ \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}. \]

can find a polynomial of degree \( n \) that goes through any \( n + 1 \)
given points in the plane).