1. Give a geometric description (e.g., as a particular line or plane) of the set of all linear combinations of the following sets of vectors.

(a) \[ \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} \]

The set of linear combinations of these vectors is the plane \(3x - z = 0\). Here’s a way to see that. (We’ll develop a more systematic approach in the next few classes.) It’s clear that each of the two vectors satisfies the equation \(3x - z = 0\), so every scalar multiple of the two vectors lies in that plane, and then every sum of scalar multiples of the two vectors lies in that plane. But the linear combinations are just the sums of scalar multiples, so the set of linear combinations of the two given vectors is contained in that plane.

To see that every vector in the plane is a linear combination of the two vectors, let \((b_1, b_2, 3b_1)\) be a vector in the plane. (Clearly every vector in the plane has that form.) Then if \(x_1 = -2b_1 + b_2\) and \(x_2 = 3b_1 - b_2\), we have

\[
\begin{bmatrix} x_1 \\ 3 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2b_1 + b_2 + 3b_1 - b_2 \\ -6b_1 + 3b_2 + 6b_1 - 2b_2 \\ -6b_1 + 3b_2 + 9b_1 - 3b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ 3b_1 \end{bmatrix}.
\]

(b) \[ \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\} \]

The set of linear combinations of the three vectors is the plane \(y = 0\) in \(\mathbb{R}^3\). It’s clear that every linear combination of these vectors has its second component equal to 0 and so lies in that plane.

To see that every vector in \(\mathbb{R}^3\) is a linear combination of these vectors, we can solve for the scalars \(x_1, x_2, x_3\) in the system of equations associated with

\[
\begin{bmatrix} x_1 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ b_3 \end{bmatrix}
\]

where \(b_1\) and \(b_3\) can be any real numbers. This gives us the equations

\[
2x_1 + 2x_2 + 4x_3 = b_1 \\
-x_1 + x_2 + x_3 = b_3
\]
Dividing the first equation by 2 and then replacing the second equation by the sum of the first equation and the second equation, the second equation becomes 

\[ 2x_2 + 3x_3 = \frac{1}{2}b_1 + b_3. \]

We can choose any real number \( t \) for \( x_3 \). Given \( t \), we have

\[ x_2 = \frac{1}{2}b_1 + \frac{1}{2}b_3 - \frac{3}{2}t. \]

Then the first equation gives us

\[ x_1 = \frac{1}{2}b_1 - \frac{1}{2}b_3 - \frac{3}{2}t. \]

So every vector in the plane \( y = 0 \) is a linear combination of our 3 vectors in infinitely many (because there are infinitely many choices for \( t \)) ways.

2. Consider the figure below:

Let \( u = av + bw \) be some linear combination with \( a + b = 1 \) and \( a, b \geq 0 \). What can you say about where \( u \) lies in the figure?

If \( a + b = 1 \), then \( b = 1 - a \), so the linear combination is \( av + (1-a)w \). But we can rewrite that as \( v + (1-a)(v-w) \). The vector \( v - w \) is the vector running from the end (arrowhead) of \( v \) to the end of \( w \), so \( u \) is the vector from the origin to the point \( a \) of the way from the end of \( v \) to the end of \( w \). The vectors \( u \), for all values of \( a \) from 0 to 1, fill in the third side of the triangle with sides \( v \) and \( w \).

3. Let \( u = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \) and \( w = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \). Write down equations for real numbers \( c, d, e \) such that \( cu + dv + ew = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \). Solve for \( c, d, \) and \( e \).

The system of equations is:

\[
\begin{align*}
2c + 0d + 0e &= 3 \\
c + 3d + 0e &= 3 \\
-c - d - 2e &= 3
\end{align*}
\]

This is lower triangular, so it’s easy to solve. The first equation tells us that \( c = \frac{3}{2} \). Then the second equation says that \( 3d = 3 - \frac{3}{2} \), so \( d = \frac{1}{2} \). With values for \( c \) and \( d \), the third equation says \(-\frac{3}{2} - \frac{1}{2} - 2e = 3 \), so \( e = -\frac{5}{2} \).
4. Find a unit vector in the direction of \( \mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \), also find a unit vector perpendicular to \( \mathbf{u} \).

The length of \( \mathbf{u} \) is \( \Vert \mathbf{u} \Vert = \sqrt{2^2 + 1^2} = \sqrt{5} \). We know that the vector \( \frac{1}{\Vert \mathbf{u} \Vert} \mathbf{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) is a unit vector in the direction of \( \mathbf{u} \).

It’s easy to see that for \( \mathbf{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \), we have \( \mathbf{u} \cdot \mathbf{w} = 0 \), so \( \mathbf{w} \) is perpendicular to \( \mathbf{u} \). The vector \( \frac{1}{\Vert \mathbf{w} \Vert} \mathbf{w} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \) is a unit vector perpendicular to \( \mathbf{u} \) and any vector in the direction of \( \mathbf{u} \).

5. Show that the dot product satisfies \( (a \mathbf{v} + b \mathbf{w}) \cdot \mathbf{u} = a(\mathbf{v} \cdot \mathbf{u}) + b(\mathbf{w} \cdot \mathbf{u}) \) for all real number \( a \) and \( b \) and all vectors \( \mathbf{v}, \mathbf{w}, \) and \( \mathbf{u} \) in \( \mathbb{R}^n \).

Let \( \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \), and similarly for \( \mathbf{w} \) and \( \mathbf{u} \). Then \( a \mathbf{v} + b \mathbf{w} = \begin{bmatrix} av_1 + bw_1 \\ \cdots \\ av_n + bw_n \end{bmatrix} \).

From the definition of the dot product, we have \( (a \mathbf{v} + b \mathbf{w}) \cdot \mathbf{u} = (av_1 + bw_1)u_1 + (av_2 + bw_2)u_2 + \cdots + (av_n + bw_n)u_n = av_1u_1 + bw_1u_1 + av_2u_2 + bw_2u_2 + \cdots + av_nu_n + bw_nu_n \). Reordering the terms and factoring out the \( a \) and \( b \), we can write this as \( a(\mathbf{v}_1u_1 + v_2u_2 + \cdots + v_nu_n) + b(w_1u_1 + w_2u_2 + \cdots + w_nu_n) \), which is \( a(\mathbf{v} \cdot \mathbf{u}) + b(\mathbf{w} \cdot \mathbf{u}) \).

6. Let \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \). Find a matrix \( A \) such that \( A\mathbf{x} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} \).

Could you do this for any permutation of the entries in \( \mathbf{x} \)? How about in higher dimensions?

If the columns of the matrix \( A \) are \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \), we want
\[ x_1 \mathbf{a} + x_2 \mathbf{b} + x_3 \mathbf{c} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} \]. It’s clear that the matrix
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]
works.

To answer the next part of the question, we need to have some notation for “any permutation”. A permutation of the components of a vector \( \mathbf{x} \) would be a vector with the same
components, but in a possibly different order. (I say “possibly” because mathematicians usually want to include “trivial” permutation, which doesn’t change the order as a permutation. That’s important when you’re studying the algebraic structure of all the permutations.) So if the vector is \((x_1, x_2, x_3)\), the permuted vector will have the same numbers as its components, but in a possibly different order, like \((x_1, x_3, x_2)\). For this problem, it’s useful to think of matrix multiplication from the row picture, so the \(i\)-th component of the product \(Ax\) is the dot product of the \(i\)-th row of \(A\) with \(x\). If the first component of the permuted vector is \(x_i\), then we want the dot product of the first row of \(A\) with \(x\) to be \(x_i\). An easy way to get that is to take the first row of \(A\) to be the vector with a 1 in the \(i\)-th column and 0s elsewhere. We can use the same approach to get the other rows of \(A\). (Note that the matrix \(A\) we get is the one we’d obtain by taking the identity matrix, the square matrix with 1s down the main diagonal from upper left to lower right and 0s elsewhere, and permuting the rows in exactly the same way we permute the entries of \(x\).

The same approach works in exactly the same way in higher dimensions. So to find an \(n \times n\) matrix that permutes the entries of a vector \(x \in \mathbb{R}^n\) in a particular way, we take the \(n \times n\) identity matrix and permute its rows that way.