Section 6.1

2. Find the eigenvalues and eigenvectors of these two matrices:

\[ A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}. \]

\( A + I \) has the ___eigenvectors as \( A \). Its eigenvalues are ___by 1.

\[ \det(A - \lambda I) = (1 - \lambda)(3 - \lambda) - 8 = 3 - 4\lambda + \lambda^2 - 8 = \lambda^2 - 4\lambda - 5 - (\lambda - 5)(\lambda + 1). \] So the eigenvalues are \( \lambda_1 = 5 \) and \( \lambda_2 = -1 \).

Setting \( \lambda = 5 \) and we see that the nullspace of \( A - 5I \) consists of all multiples of \((1, 1)\), so the eigenvectors with eigenvalue 5 are those multiples. And the nullspace of \( A + I \) consists of all multiples of the vector \((-2, 1)\), so those are the eigenvectors with eigenvalue -1.

If \( Av = \lambda v \), then \( (A + I)v = \lambda v + v = (\lambda + 1)v \). So the eigenvalues of \( A + I \) are \( 5 + 1 = 6 \) and \(-1 + 1 = 0 \). But the eigenvectors of \( A + I \) are the same as the eigenvectors of \( A \).

3. Compute the eigenvalues and eigenvectors of \( A \) and \( A^{-1} \). Check the trace.

\[ A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & 0 \end{bmatrix}. \]

\( A^{-1} \) had the ___eigenvectors as \( A \). When \( A \) has eigenvalues \( \lambda_1 \) and \( \lambda_2 \), its inverse has eigenvalues ___.

There are a couple of ways to see that the eigenvectors of \( A \) are 2 and -1. The straightforward way is to observe that \( \det(A - \lambda I) = \lambda(1 - \lambda) - 2 = (\lambda + 1)(\lambda - 2) \), but you can also observe that the determinant of \( A \) (the product of the eigenvalues) is -2 and the trace of \( A \) (the sum of the eigenvalues) is 1. So the product of the two eigenvalues is -2 and their sum is 1. Corresponding eigenvectors are \((1, 1)\) and \((2, -1)\), respectively.

For \( A^{-1} \), we can compute the eigenvalues and eigenvectors directly, but it’s better to think about what they really mean. If \( v \) is an eigenvector of \( A \) with eigenvalue \( \lambda \), then \( Av = \lambda v \). So \( v = A^{-1}Av = A^{-1}(\lambda v) = \lambda A^{-1}v \). It follows that \( A^{-1}v = \frac{1}{\lambda}v \). (Note that all eigenvalues of an invertible matrix must be nonzero, since the determinant is the product of the eigenvalues.) So if \( v \) is an eigenvector of \( A \) with eigenvalue \( \lambda \), \( v \) is also
an eigenvector of $A^{-1}$ with eigenvalue $\frac{1}{\lambda}$. Thus, $A^{-1}$ has the same eigenvectors as $A$ and its eigenvalues are the reciprocals of the eigenvalues of $A$.

7. Elimination produces $A = LU$. The eigenvalues of $U$ are on its diagonal; they are the $\lambda$. The eigenvalues of $L$ are on its diagonal; they are all $1$. The eigenvalues of $A$ are not the same as $\lambda$.

The eigenvalues of $U$ are the pivots (together with 0, if $A$ is singular). The eigenvalues of $L$ are all 1. The eigenvalues of $A$ are not (necessarily) the same as the pivots.

9. What do you do to the equation $Ax = \lambda x$, in order to prove (a), (b), and (c)?

(a) $\lambda^2$ is an eigenvalue of $A^2$, as in Problem 4.

If $Ax = \lambda x$, then you multiply both sides by $A$: $A^2x = A(Ax) = A(\lambda x) = \lambda (Ax) = \lambda^2 x$.

(b) $\lambda^{-1}$ is an eigenvalue of $A^{-1}$, as in Problem 3.

You multiply both sides by $A^{-1}$: $A^{-1}(Ax) = A^{-1}(\lambda x)$ as in the solution to Problem 3 above.

(c) $\lambda + 1$ is an eigenvalue of $A + I$, as in Problem 2.

You add $Ix = x$, as in the solution to Problem 2 above.

16. The determinant of $A$ equals the product $\lambda_1 \lambda_2 \ldots \lambda_n$. Start with the polynomial $\det(A - \lambda I)$ separated into its $n$ factors (always possible [at least when working over the complex numbers]). The set $\lambda = 0$:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$$

so $\det A = \lambda$. Check this rule in Example 1 where the Markov matrix has $\lambda = 1$ and $\frac{1}{2}$.

If we set $\lambda = 0$ in the factored expression for $\det(A - \lambda I)$, we get $\det(A) = \det(A - 0I) = \lambda_1 \lambda_2 \ldots \lambda_n$.

In Example 1, the determinant of $A$ is $(.8)(.7) - (.2)(.3) = .5$ and the eigenvalues are 1 and .5.

Section 6.2

1. (a) Factor these two matrices into $A = XAX^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$
For the first matrix, the eigenvalues are obviously 1 and 3, with corresponding eigenvectors \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), respectively.

So we have \( X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) with \( X^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \) and \( \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \). You can check that \( A = XAX^{-1} \).

We see that the second matrix is singular since the second row is a multiple of the first row. So the determinant is 0. Since the trace is 4, we know that one eigenvalue is 0 and the other is 4. (Of course, you can do this by taking the determinant of \( A - \lambda I \), too.) An eigenvector with eigenvalue 0 is \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) and an eigenvalue with eigenvector 4 is \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \). So we can take \( X = \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix} \) with \( X^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \) and \( \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \). Again, you can check that \( A = XAX^{-1} \).

(b) If \( A = XAX^{-1} \) then \( A^3 = (XAX^{-1})(XAX^{-1})(XAX^{-1}) = X(A^3)X^{-1} \) and, using the fact that the inverse of a product is the product of the inverses in reverse order, \( A^{-1} = X\Lambda^{-1}X^{-1} \).

2. If \( A \) has \( \lambda_1 = 2 \) with eigenvector \( x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \lambda_2 = 5 \) with eigenvector \( x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), use \( A = X\Lambda X^{-1} \) to find \( A \). No other matrix has the same \( \lambda \)'s and \( x \)'s.

The diagonal matrix with the eigenvalues on the diagonal is \( \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \). We can take \( X \) to be \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \), the matrix with the eigenvectors as its columns. Then \( X^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \) and we have \( A = X\Lambda X^{-1} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \).

3. Suppose \( A = X\Lambda X^{-1} \). What is the eigenvalue matrix for \( A + 2I \)? What is the eigenvector matrix? Check that \( A + 2I = (X\Lambda)(X^{-1})^{-1} \).

The eigenvalue matrix for \( A + 2I \) is \( \Lambda + 2I \). The eigenvector matrix is \( X \). We have \( X(\Lambda + 2I)X^{-1} = X\Lambda X^{-1} + X(2I)X^{-1} = A + 2I \).

Section 8.1
3. Which of these transformations are not linear? The input is $v - (v_1, v_2)$.

(a) $T(v) = (v_2, v_1)$

This is linear. It can be represented by the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so $T(v) = Av$. We know that any transformation defined by multiplication by a matrix this way must be linear.

(b) $T(v) = (v_1, v_1)$

This is linear, with standard matrix $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

(c) $T(v) = (0, v_1)$

This is linear, with standard matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

(d) $T(v) = (0, 1)$

This is not linear. For instance $T((0, 0) + (0, 0)) = (0, 1)$ but $(T((0, 0)) + T((0, 0)) = (0, 1) + (0, 1) = (0, 2)$

(e) $T(v) = v_1 - v_2$

This is linear, with matrix $\begin{bmatrix} 1 & -1 \end{bmatrix}$.

(f) $T(v) = v_1 v_2$

This is not linear. $T((1, 2)) = 2$ but $T(3(1, 2)) = T((3, 6)) = 18 \neq 3 \cdot T((1, 2))$.

4. If $S$ and $T$ are linear transformations, is $T(S(v))$ linear or quadratic?

(a) (Special case) If $S(v) = v$, then $T(S(v)) = v$ or $v^2$?

We have $T(S(v)) = T(v)$

(b) (General case) $S(v_1 + v_2) = S(v_1) + S(v_2)$ and $T(v_1 + v_2) = T(v_1) + T(v_2)$ combine into $T(S(v_1 + v_2) = T(S(v_1) + T(v_2))$, so the composition $T \circ S$ satisfies the additive condition for linearity. (It’s easy to see that it also satisfies the scalar multiplication condition.)

6. Which of these transformations satisfy $T(v + w) = T(v) + T(w)$ and which satisfy $T(cv) = cT(v)$?

(a) $T(v) = \frac{v}{\|v\|}$

This doesn’t satisfy either condition. (And it’s not even defined if $v = 0$.) Note that $T(v)$ is a unit vector in the direction of $v$. A unit vector in the direction of a sum $v_1 + v_2$ is not the sum of unit vectors in the direction of $v_1$ and $v_2$. And if we take a unit vector in the direction of $2v$, we get the same thing as if we take a unit vector in the direction of $v$, not 2 times that unit vector.
(b) \( T(\mathbf{v}) = v_1 + v_2 + v_3 \)

This \( T \) is a linear transformation with matrix \[
\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.
\]

(c) \( T(\mathbf{v}) = (v_1, 2v_2, 3v_3) \)

This \( T \) is also a linear transformation, with matrix \[
\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}.
\]

(d) \( T(\mathbf{v}) = \) largest component of \( \mathbf{v} \)

This doesn’t satisfy either condition. Observe that \( T((1,0)+(0,1)) = 1 \neq T((1,0)+T((0,1)). \) And \( T((-1)(1,0)) = T((-1,0)) = 0 \neq -T((1,0)) = -1. \)

12. Suppose \( T \) transforms \((1,1)\) to \((2,2)\) and \((2,0)\) to \((0,0)\). Find \( T(\mathbf{v}) \):

(a) \( \mathbf{v} = (2,2) \)

Since \( (2,2) = 2(1,1) \), we know that \( T((2,2)) = 2T((1,1)) = 2(2,2) = (4,4). \)

(b) \( \mathbf{v} = (3,1) \)

\[
(3,1) = (1,1)+(2,0) \text{ so } T((3,1)) = T((1,1))+T((2,0)) = (2,2).
\]

(c) \( \mathbf{v} = (-1,1) \)

\[
(-1,1) = (1,1)-(2,0) \text{ so } T((-1,1)) = T((1,1))-T((2,0)) = (2,2).
\]

(d) \( \mathbf{v} = (a,b) \)

\[
(a,b) = b(1,1)+\frac{a-b}{2}(2,0) \text{ so } T((a,b)) = b(2,2)+\frac{a-b}{2}(0,0) = b(2,2).
\]

13. \( M \) is any \( 2 \times 2 \) matrix and \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \). The transformation \( T \) is defined by \( T(M) = AM \). What rules of matrix multiplication show that \( T \) is linear?

The distributive property (page 73 in the text) says that \( A(M_1+M_2) = AM_1+AM_2 \), so \( T(M_1+M_2) = T(M_1)+T(M_2). \)

For scalar multiplication, we need the property that \( A(cM) = (cA)M = c(AM). \) I don’t think the textbook gives this a name, but the definition of matrix multiplication as the dot product of a row and a column implies this property (since \( (cv) \cdot \mathbf{w} = c(v \cdot \mathbf{w}). \)

14. Suppose \( A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \). Show that the range of \( T \) is the whole matrix space \( \mathbf{V} \) and the kernel is the zero matrix:

(a) If \( AM = 0 \), prove that \( M \) must be the zero matrix.

The matrix \( A \) is invertible. If \( AM = 0 \), then \( A^{-1}AM = IM = M \) must also be 0.
(b) Find a solution to $AM = B$ for any $2 \times 2$ matrix $B$.

The solution is $M = A^{-1}B$, since $A(A^{-1}B) = B$. In this case, $A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

8.2

1. The transformation $S$ takes the second derivative. Keep $1, x, x^2, x^3$ as the input basis $v_1, v_2, v_3, v_4$ and also as the output basis $w_1, w_2, w_3, w_4$. Write $S(v_1), S(v_2), S(v_3), S(v_4)$ in terms of the $w$'s. Find the $4 \times 4$ matrix $A_2$ for $S$.

So $S(v) = \frac{d^2 v}{dx^2}$. Then $S(v_1) = S(v_2) = 0, S(v_3) = 2v_1,$ and $S(v_4) = 6v_2$. The matrix for $S$ with respect to these bases is

$$
\begin{bmatrix}
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

5. With bases $v_1, v_2, v_3,$ and $w_1, w_2, w_3,$ suppose $T(v_1) = w_2$ and $T(v_2) = T(v_3) = w_1 + w_3$. $T$ is a linear transformation. Find the matrix $A$ and multiply by the vector $(1, 1, 1)$. What is the output from $T$ when the input is $v_1 + v_2 + v_3$?

The matrix $A$ is

$$
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 1
\end{bmatrix}.
$$

Multiplying $A$ by $(1, 1, 1)$ gives $(2, 1, 2)$, and the output from $T$ when the input is $v_1 + v_2 + v_3$ is $T(v_1) + T(v_2) + T(v_3) = 2w_1 + w_2 + 2w_3$.

6. Since $T(v_2) = T(v_3)$, the solutions to $T(v) = 0$ are $v = \underline{\phantom{0}}$. What vectors are in the nullspace of $A$? Find all solutions to $T(v) = w_2$.

Since $T(v_2 - v_3) = T(v_2) - T(v_3) = 0$, there are nonzero solutions to $T(v) = 0$, and hence infinitely many solutions. The nullspace of $A$ consists of the vectors of the form $(0, c, -c)$. The solutions to $T(v) = w_2$ are all the vectors of the form $v_1 + c(v_2 - v_3)$.

8. You don’t have enough information to determine $T^2$. Why is its matrix not necessarily $A^2$? What more information do you need?

To determine $T^2$, we’d need to know the $T(w_i)$, and we’re only given the $T(v_i)$ (as linear combinations of the $w_i$). Of course, if the bases are the same (with $v_i = w_i$ for $i = 1, 2, 3$), then the matrix for $T^2$ is just $A^2$. 


10. Suppose \( T \) (assumed to be invertible) with \( T(v_1) = w_1 + w_2 + w_3 \) and 
\( T(v_2) = w_2 + w_3 \) and \( T(v_3) = w_3 \). Find the matrix \( A \) for \( T \) using these 
basis vectors. What input vector \( v \) gives \( T(v) = w_1 \)?

The matrix \( A \) is 
\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}
\]

Since \( A \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), we see 
that \( T(v_1 - v_2) = w_1 \).

14. (a) What matrix \( B \) transforms \((1, 0)\) into \((2, 5)\) and transforms \((0, 1)\) into 
\((1, 3)\)?

\[
\begin{bmatrix}
2 & 1 \\
5 & 3
\end{bmatrix}
\]

(b) What matrix \( C \) transforms \((2, 5)\) to \((1, 0)\) and \((1, 3)\) to \((0, 1)\)?

\[
C = B^{-1} = \begin{bmatrix}
3 & -1 \\
-5 & 2
\end{bmatrix}
\]

(c) Why does no matrix transform \((2, 6)\) to \((1, 0)\) and \((1, 3)\) to \((0, 1)\)?

For any matrix \( A \), \( A \begin{bmatrix} 2 \\ 6 \end{bmatrix} = 2A \begin{bmatrix} 1 \\ 3 \end{bmatrix} \). (Multiplication by a 
matrix is a linear transformation.) So no matrix can trans-
form \((2, 6)\) to \((1, 0)\) and \((1, 3)\) to \((0, 1)\).