The format of the exam will be similar to those of the first two exams with a mix of short answer problems, computations, and questions where you have to explain why something is true or false. (But note that this is a 2-hour exam, no 75 minutes.) As before, the homework is a good general guide to the kind of problems you’ll see, though some of the homework problems are probably too long for an in-class exam. So make sure you have gone through the posted solutions to homework problems. You’ll have the whole class period (but people who come late won’t get extra time without a very good reason). No notes, books, or electronic devices can be used during the exam.

The exam is cumulative, so there will be some questions from the part of the course covered by the first two exams, but more weight will be given to the material from the last part of the course: Sections 3.5, 4.1, 4.2, 4.3, 4.4, 5.1, 5.2, 5.3, 6.1, 6.21, 8.1, and 8.2. You’re responsible for all the material in the textbook in those sections (together with the material listed on the review sheets for the first two exams), whether we discussed it in class or not.

You should go over the review sheets for the first two exams, which are still available on the course web site if you need them. Here is a bit more detail on the new topics for the exam. Keep in mind that not everything listed here will be on the exam and that something not being mentioned explicitly here doesn’t mean it won’t be on the exam.

- **Section 3.5:** The key idea here is the connections between the dimensions of the various subspaces associated with a matrix, e.g., that the sum of the dimensions of the nullspace and the column space is equal to the number of columns, and that the dimension of the row space (or the column space of the transpose) is equal to the dimension of the column space.

- **Chapter 4:** You should understand what it means for two subspaces to be orthogonal and what the orthogonal complement of a subspace is. Note that dealing with orthogonal complements makes use of the following properties of bases: If a subspace has dimension $d$, then any set of $d$ independent vectors in the subspace must be a basis for the subspace, and any set of $d$ vectors that spans the subspace must be a basis for the subspace.

Using these notions of orthogonality, we defined the orthogonal projection of a vector onto a subspace, and observed that this projection is the element in the subspace that is closest to the original vector. Given a subspace (perhaps by a basis, or by some other way of defining the subspace), you should be able to compute the orthogonal projection of any vector onto the subspace, and to construct the projection matrix that multiplies any vector to its orthogonal projection. As an application of this, you should be able to compute the best “least squares” approximation to a
solution of $A \mathbf{x} = \mathbf{b}$ when the equation doesn’t have any solutions—this is the vector $\mathbf{x}$ that makes $A \mathbf{x}$ as close as possible to $\mathbf{b}$, and $A \mathbf{x}$ is the orthogonal projection of $\mathbf{b}$ onto the column space of $A$.

The projection matrix and the least squares approximation, are much easier to compute when we have an orthonormal basis for the subspace. You should know why, and how to get an orthonormal basis from any basis (the Gram-Schmidt process). You should also know some basic facts about orthogonal matrices, e.g., that if $Q$ is orthogonal, $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ and $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$. (We say that “orthogonal matrices preserve length and dot products (and therefore angles between vectors)”.

• **Chapter 5:** The key notion here is that the determinant, as a function from $n \times n$ matrices to $\mathbb{R}$, is the unique function that takes the value 1 on the identity matrix, changes sign if you interchange two rows (or columns) of the matrix, and is “linear” in each row (in the sense explained by Strang). Since all the different ways of thinking of the determinant satisfy these properties (and the other 7 or so properties that Strang derives from these), all those ways of thinking about the determinant must give the same function. So you need to know the properties of the determinant (the three key ones, plus the ones Strang derives), and the three standard ways of thinking about the determinant: the “big formula”, which considers all the ways of taking a product of entries in the matrix that includes exactly one entry from each row and column, the product of the pivots when we do elimination, and the recursive expansion by cofactors. And also the (signed) volume of the box (or polytope) formed by the columns (or rows) of the matrix.

The cofactor matrix gives us a formula for the inverse of a matrix, though this is of mostly theoretical use, since in practice if we need the inverse, we’re better off just doing Gauss-Jordan elimination to find it. It also gives us Cramer’s rule for solving a linear system, which is also of primarily theoretical use. But some of the theoretical uses are important and, very occasionally, we know enough about a matrix to make the cofactor matrix and Cramer’s rule useful in practice. So you need to know these things.

• **Chapter 6, Eigenvalues:** We don’t have time in this course to really explain why eigenvalues and eigenvectors are important, so you’ll mostly have to take my word (and Strang’s) for it until you see the applications in another course. What you need to know is the definitions and how to compute the eigenvalues (at least when you can find the roots of the polynomial $\det(A - \lambda I) = 0$).

When an $n \times n$ matrix $A$ has $n$ independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, the matrix $X^{-1}AX$ is diagonal, where the columns of $X$ are the $\mathbf{x}_i$. Diagonal matrices are vastly easier to work with than general matrices (so this gives some hint as to why eigenvectors and eigenvalues are important), and I
told you a little bit about how we can use this to solve systems of differential equations (or to solve single higher-order differential equations by converting them to systems). You are not responsible for the connection with differential equations, but you do need to know about the “diagonalization” using the independent eigenvectors. (I also showed you what happens when you don’t have \( n \) independent eigenvectors, where you get a block matrix with the eigenvalues on the diagonal and 1s in some of the entries just below the diagonal; you don’t need to know that for the exam, either.)

- **Chapter 8:** In each part of mathematics, there is a certain class of functions that “make sense” for what we’re trying to do. For calculus, for instance, those are the differentiable functions. In linear algebra, we’re interested in functions from one vector space to another, and the ones that properly respect the vector space operations are the linear transformations. You need to know the definition of linear transformation and be able to apply it in various settings.

It’s not an accident that most of the examples of linear transformations have the form \( T(x) = Ax \), where \( A \) is a matrix. In fact, every linear transformation can be represented by a matrix, where the matrix depends on the coordinates we use to describe the input and output vectors. In a vector space, those coordinates depend on choosing a basis: Given a basis \( x_1, \ldots, x_n \), we know that every vector \( v \) can be written as a linear combination \( \sum_{i=1}^{n} a_i v_i \) and the \( a_i \) uniquely determine the vector. So if we know the basis, we can describe the vector \( v \) by just listing the \( a_i \), for instance as \( [a_1, \ldots, a_n] \). So you need to know how this works, and how the bases for the input and output spaces affect the resulting matrix. (For instance, if we have a basis consisting of eigenvectors for a given linear transformation from a vector space to itself, the matrix with respect to that basis (used for both the input and output) will be diagonal.) For the special case of a linear transformation from a space \( V \) to itself, you should understand how the matrix \( A \) with respect to the standard basis is related to the matrix with respect to a basis \( B \), \( (B^{-1}AB) \), where \( B \) gives the vectors of \( B \) in standard coordinates. (This is how we diagonalize an \( n \times n \) matrix that has \( n \) independent eigenvectors.)

Lots of linear algebra (which you’ll see in later courses that apply it) depend on choosing the right bases for a particular application.