Math 235H — Spring 2017
Review Sheet for Exam 1

I will have extra office hours on Monday, February 27, at 2:00 and Wednesday, March 1, at 2:30, as well as my regular office hour on Tuesday at 10:00. (I will have to cancel my Friday office hour on March 3, because I have to teach a class for another faculty member who is away.)

The exam will be a mix of short-answer questions, calculations, and questions where you might be asked to explain why something is true or false. By “short-answer questions”, I mean things like definitions, true-false or multiple-choice questions, or problems that ask you to give an example or non-example of something. The homework is a good general guide to the kind of problems you’ll see, though some of the homework problems are probably too long for an in-class exam. So make sure you have gone through the posted solutions to homework problems.

You’ll have the whole class period (but people who come late won’t get extra time without a very good reason). No notes, books, or electronic devices can be used during the exam.

The exam will cover the material in the textbook from Chapter 1 and Chapter 2 through Section 2.5. You’re responsible for all the material in the textbook in those sections, whether we discussed it in class or not.

Here is a bit more detail on the topics for the exam. Keep in mind that not everything listed here will be on the exam and that something not being mentioned explicitly here doesn’t mean it won’t be on the exam.

- Vectors and basic operations: You should know what we mean by a vector (as an ordered list of numbers) and the notation using columns or rows. You should understand the basic operations of addition and scalar multiplication both algebraically and (at least in 2 or 3 dimensions) geometrically. You should understand what a linear combination of some vectors is. One of the main things we’ll be focusing on throughout the course is describing all the linear combinations of a set of vectors. You should understand the definition of the dot product of two vectors (that have the same number of components) and the geometric interpretations in terms of lengths and angles, in particular that two vectors are perpendicular exactly when their dot product is 0.

- Matrices: We initially discussed matrices as rectangular arrays of numbers, basically as a way of combining vectors into a single object so that we could represent a system of linear equations in the form $Ax = b$, where $A$ is the matrix of coefficients. For this, we defined the multiplication of a matrix (with $n$ columns) times a vector (with $n$ components or rows). The discussion of matrix multiplication in Chapter 1 is really intended only to provide some motivation and examples for what comes later, but the idea of the product $Av$ being a linear combination of the columns of
v (the one given by the components of the vector v) is important. (The book also talks about inverses in Section 1.3 without really defining them; you should think about them in the way we discussed them in Chapter 2. The book also briefly talks about dependence and independence in 3 dimensions in Section 1.3, but I won’t ask you about that until we get to a more general treatment later in the course.)

• Elimination: As we discussed, the way we solve a system of linear equations is to (more or less systematically) transform it into another system that has exactly the same solutions as the original one and from which we can easily find those solutions. Our basic approach is to use two kinds of operations to transform our system to one for which the coefficient matrix is upper-triangular. This amounts to eliminating the i-th variable from all equations after the i-th one, by subtracting multiples of the i-th equation from the following ones. (But sometimes we need to interchange equations to make this work.) This idea is introduced in the book in 2 and 3 dimensions, but applies in n dimensions. Once we have the upper-triangular system with the same solutions as our original system, we can solve by back substitution, starting with the last equation and working back to the first. If we get an equation of the form 0 = b, where b ≠ 0, we know the system has no solutions. The nonzero numbers on the diagonal in the resulting upper-triangular matrix are called the pivots and the number of pivots is an important quantity. In particular, if there are columns that don’t have pivots, our system Ax = b will have infinitely many solutions for some values of b.

• Matrix multiplication: The idea of multiplying a matrix and a vector was introduced in Chapter 1. In Chapter 2, this is more carefully extended to the product of an m × n matrix and an n × p matrix, yielding an m × p matrix. We defined this product by setting the (i, j) entry of the product matrix AB be the dot product of the i-th row of A with the j-th column of B. But it’s important to understand other ways to think of this product matrix as well: (i) the j-th column of AB is the linear combination of the columns of A with the coefficients coming from the j-th column of B (so the columns of AB are A times the columns of B); (ii) the i-th row of AB is the linear combination of the rows of B with the coefficients coming from the i-th row of A (so the rows of AB are the rows of A times B); (iii) the columns of A are m × 1 matrices and the rows of B are 1 × p matrices, and the product of an m × 1 matrix and and a 1 × p matrix is an m × p matrix—the matrix product AB is the sum of the n matrices formed by taking the product of the i-th column of A and the i-th row of B, as i goes from 1 to n. You can check that these all give the same value for AB as our dot product definition by just writing out the calculations. These different views of matrix multiplication can be useful in different contexts. You should also know about addition of matrices of the same size and the basic properties of the matrix operations, such as the associative property of matrix multiplication, the distributive laws,
etc. Keep in mind that matrix multiplication is not commutative: for most matrices $A$ and $B$, even if both $AB$ and $BA$ are defined, they won’t be equal. You should understand how block multiplication works for matrices and blocks of appropriate sizes (just check that the relevant dot products work out to be the same).

- Once we have these ways of thinking about matrix multiplication, we can think of elimination as a process of multiplying the augmented matrix of a system by some special types of matrices, the ones that give a product that represents subtracting a multiple of the $i$-th row to the $j$-th row (for some $j > i$) and the ones that interchange two rows. This isn’t so important for solving a single system of equations, where we can just do elimination on the augmented matrix directly, but it’s important for understanding when matrices have inverses and for applying deeper insights about systems of linear equations in a wide variety of contexts, especially when we come to think about matrices as representing functions.

- Inverses: The discussion of inverses applies only to square matrices—it turns out to be important that we can multiply $A$ and $A^{-1}$ in either order, and this only makes sense if they’re square matrices of the same size. (This is connected to the idea of a matrix representing a function, where we want an invertible matrix to represent a function with an inverse function where we can compose the functions in either order.) So assume $A$ is an $n \times n$ matrix. You should know the definition of an inverse for $A$, why having an inverse means that every system of equations $Ax = b$ has exactly one solution, which also means that when we do elimination we’ll get $n$ pivots. (The definition of determinant of a square matrix introduced in section 2.5 is just the product of the diagonal entries after doing elimination, so having fewer than $n$ pivots is the same as saying that one of the diagonal entries, and hence the determinant, is 0. I won’t ask you to do anything with the determinant on the exam.) Thinking of matrix multiplication $AA^{-1}$ in terms of the $j$-th column in the product being the product of $A$ and the $j$-th column of $A^{-1}$, we see that we find the $j$-th column of $A^{-1}$ by solving the system of equations $Ax = \begin{bmatrix} j \text{-th column of } I \end{bmatrix}$. We saw how we can do this with the same row operations as elimination together with the operation of multiplying a row by a nonzero number, organized as Gauss-Jordan elimination on the augmented matrix, and that we can do this all at once for all the columns of $A^{-1}$ by doing Gauss-Jordan elimination on the augmented matrix $[AI]$. You should understand the examples in the book, but I won’t expect you to remember, e.g., the inverse of the matrix $K$. 