Section 12.1

1. You have a 5-foot-by-7-foot rectangular rug in your classroom. You also have a bunch of square-foot tiles and some tape measures.

   (a) What is the most primitive way to determine the area of the rug?

       The most primitive way would be to cover the entire rug with (nonoverlapping) square-foot tiles, and count the tiles used.

   (b) What is a less primitive way to determine the area of the rug and why does this method work?

       One would be to use the tiles to cover two edges and the corner at which they meet (using a total of 11 tiles, since the corner tile counts for both edges), and use this information to observe that there would be 5 rows of 7 tiles if you covered the entire rug.

3. (a) Explain how to decompose the large rectangle in Figure 12.4 into $2 \frac{1}{2}$ groups with $3 \frac{1}{2}$ squares in each group so as to describe the area of the rectangle as $2 \frac{1}{2} \cdot 3 \frac{1}{2} \text{ cm}^2$.

       If we view each row as a group, there will be $2 \frac{1}{2}$ equal groups with $3 \frac{1}{2}$ squares in each group.

   (b) Calculate $2 \frac{1}{2} \cdot 3 \frac{1}{2}$ without a calculator, showing your calculations. Then verify that this calculation has the same answer as when you determine the area of the rectangle in Figure 12.4 by counting squares.

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       2 \frac{1}{2} \cdot 3 \frac{1}{2} = \frac{5}{2} \cdot \frac{7}{2} = \frac{35}{4} = 8 \frac{3}{4}.
       \]

5. (a) Draw a (fairly long) line segment and designate it as being 1 unit long. Then draw a 0.6-unit-by-0.9-unit rectangle.

       I leave the drawing to you.

   (b) apply the length-width formula for the area of the rectangle and verify that the formula gives you the correct area for your rectangle in part (a). Attend carefully to units of area.

       The area of the rectangle is $(0.6) \cdot (0.9) = 0.54 \text{ square units}$. (If you think of dividing the rectangle up into squares, you'd probably use squares that are 0.1 units-by-0.1 units, and thus have an area of 0.01 square units. There would be 54 such squares in the rectangle, so 0.54 square units.)
(c) When you applied the length-width formula to find the area of the rectangle in part (b), you used lengths of 0.6 and 0.9 units. Describe those lengths and show them in your drawing.

The 0.6 units is the length of a column (or row, depending on how you drew it) of 0.1 units-by-0.1 units squares along one edge of the rectangle. Note that students may be confused about what the real units are here.

Section 12.2

2. Figure 12.14 shows the floor plan for a one-story house. Calculate the area of the floor of the house, explaining your reasoning.

You can think of this as a 24-by-40 rectangle on top of both a 16-by-24 rectangle and a 32-by-40 rectangle. Or as a 56-by-64 rectangle with two rectangular corners (16-by-24 and 24-by-24) removed. Either way, you should get 2624 square feet.

3. An area problem: The Johnsons are planning to build a 5-foot-wide brick walkway around their rectangular garden, which is 20 feet wide and 30 feet long. What will the area of the walkway be? Before you solve the problem yourself, use Kaitlyn’s idea.

   (a) Kaitlyn’s idea is to “take away the area of the garden.” Explain how to solve the problem about the area of the walkway by using this idea. Explain clearly how to apply one or both of the moving and additivity principles on area in this case.

      Kaitlyn’s idea is that, after the walkway is built, the outside of the walkway will form a rectangle that has dimensions 10 feet more than the dimensions of the garden (the walkway adds 5 feet on each side). So that rectangle has an area of $30 \cdot 40 = 1200$ ft$^2$. The area of the garden is $20 \cdot 30$ ft$^2$, so the walkway will have an area of 600 ft$^2$.

   (b) Now solve the problem about the area of the walkaway in another way than you did in part (a). Explain your reasoning.

      There are several ways to do this. One is to divide the walkway into 4 rectangles: 2 with dimensions 40 \cdot 5 and 2 with dimensions 20 \cdot 5 (you don’t want to double count the corners, so it’s not 30 \cdot 5). That gives 600 square feet again. Or you could take 4 corners at 5 ft-by-5 ft, 2 20-by-5 rectangles, and 2 30-by-5 rectangles.

4. Figure 12.15 shows a design for an herb garden, with approximate measurements. Four identical plots of land in the shape of right triangles (shown lightly shaded) are surrounded by paths (shown darkly shaded). Use the moving and additivity principles to determine the area of the paths.
This is a trick question: with the exact dimensions shown, the triangles aren’t exactly right triangles, and they each have area $\sqrt{1305} \approx 144.5$ square feet. But you’d need to use Heron’s formula for the area of a triangle in terms of its sides to find that, and you aren’t expected to know Heron’s formula. Beckmann expects you to treat the triangles as approximately right triangles that can be put together in a square 12 feet on each side. The problem does say that the measurements shown are approximate.)

The overall area of the square region is $20 \cdot 20 = 400$ square feet. The area of the actual garden is made up of 4 triangular plots; the triangles can be put together to form a 12-by-12 square, so the area of the actual garden is 144 square feet and the area of the path is $400 - 144 = 256$ square feet.

Section 12.3

1. Use the moving and additivity principles to determine the area (in square units) of the triangle in Figure 12.30 in two different ways. Do not use a formula for areas of triangles. The grid lines in Figure 12.30 are 1 unit apart. Explain your reasoning.

We can divide the triangle into two right triangles, one with legs (the edges making the right angle) of 4 and 5 units, and the other with legs of 3 and 5 units. Duplicating each of these triangles by flipping them over their hypotenuses would give us a rectangle 5 units high by 7 units wide. That rectangle has twice the area of our original triangle (since it has 2 of each of the right triangles). So the area of the original triangle is $\frac{1}{2}$ of 35, or 17$\frac{1}{2}$ square units.

For a second way, we can use the first method of Practice Exercise 1. If we cut the triangle by a horizontal line 2$\frac{1}{2}$ units below the top edge, we get a trapezoid and a triangle. Dividing that triangle into two right triangles with a vertical line through the bottom vertex (see Figure 12.27(a) for a related picture with somewhat different dimensions and turned upside-down), we can move those triangles to make a rectangle 7 by 2$\frac{1}{2}$ units, with area $7 \cdot 2\frac{1}{2} = 17\frac{1}{2}$ square units.

2. For each triangle in Figure 12.31, show the height of the triangle that corresponds to the indicated base $b$. The use these bases and heights to determine the area of each triangle.

For triangle $A$, $h$ is 3 and the area is $\frac{1}{2}(2 \cdot 3) = 3$ cm$^2$. For triangle $B$, $h$ is 2 and the area is 1 cm$^2$. For triangle $C$, $h$ is 5 and the area is 5 cm$^2$. For triangle $D$, $h$ is 2 and the area is 3 cm$^2$. 

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6. Becky was asked to divide a rectangle into 4 equal pieces and to shade one of those pieces. Figure 12.34 shows her solution. Is Becky right or not? Explain your answer.

It depends on how you interpret the “equal”. She has correctly divided the rectangle into four triangles of equal area. Suppose the rectangle has vertical side of length \( w \) and horizontal side of length \( l \). The shaded triangle has a vertical side of the rectangle as its base and height equal to half of \( l \), so its area is \( \frac{1}{4}wl \), and the same is true for the triangle whose base is the right side of the rectangle. The top and bottom triangles have base \( l \) and height \( \frac{1}{2}w \), so the same area. But these triangles are not congruent (and they don’t have the same perimeters), so in another sense they’re not equal.

7. Explain how to use the additivity principle to determine the area of the darkly shaded triangle that is inside a rectangle in Figure 12.35.

The upper left triangle has area 5 square units, the lower left triangle has area 7 square units, and the upper right triangle has area 4 square units. The rectangle has area 28 units, so the darkly shaded triangle has area \( 28 - 5 - 7 - 4 = 12 \) square units.

Section 12.6

3. A large running track is constructed to have straight sections and two semicircular sections with dimensions given in figure 12.82. Assume that runners always run on the inside line of their lane. A race consists of one full counterclockwise revolution around the track plus an extra portion of a straight segment, to finish up at the finish line shown. What should the distance \( x \) between the two starting blocks be in order to make a fair race? Explain your reasoning.

We need to figure out the difference in the distance around the track at the inside of each lane. In each case, this is two 100-meter straight segments and two semicircles (or one circle). For lane 1, the inside lane, the radius of the circle is 40 meters, so the circumference of the circle is \( 80\pi \) and the total distance around the track is \( 200 + 80\pi \) meters. For lane 2, the outside lane, the radius is 45 meters, so the circumference of the circle is \( 90\pi \) and the total distance is \( 200 + 90\pi \) meters. The difference is \( 10\pi \) meters, so to make sure that both runners have to cover the same distance, the starting block for lane 1 should be \( 10\pi \) meters farther from the finish line.

6. Lauriann and Kinsey are in charge of the annual pizza party. In the past, they’ve always ordered 12-inch-diameter round pizzas, and each 12-inch pizza has always served 6 people. This year, the jumbo 16-inch-diameter
round pizzas are on special, so Lauriann and Kinsey decide to get 16-inch pizzas instead. They think that since a 12-inch pizza serves 6 (which is half of 12), a 16-inch pizza should serve 8 (which is half of 16). But when Lauriann and Kinsey see a 16-inch pizza, they think it should serve even more than 8 people. Suddenly Kinsey realizes the flaw in their reasoning that a 16-inch pizza should serve 8. Kinsey has an idea for determining how many people a 16-inch pizza will serve. What mathematical reasoning might Kinsey be thinking of, and how many people should a 16-inch pizza serve if a 12-inch pizza serves 6? Explain your answers.

The area of the 16-inch pizza is $\pi(8)^2 = 64\pi$, while the area of the 12-inch pizza is $\pi(6)^2 = 36\pi$. If a 12-inch pizza serves 6 people, each person gets about $6\pi$ square inches of pizza. Then a 16-inch pizza will feed about $64\pi \div 6\pi = 10\frac{2}{3}$ people.