Section 8.4

1. For which counting numbers, \( N \), greater than 1, is there only one rectangle whose side lengths, in inches, are counting numbers and whose area, in square inches, is \( N \)? Explain.

   The number \( N \) must be prime, and the rectangle will have dimensions 1 by \( N \). If we take a number \( M \) that is not prime, it could be factored as \( M = 1 \cdot M \) and \( M = A \cdot B \) for some counting numbers \( A \) and \( B \), both of which are smaller than \( M \). But then there are two different rectangles that have area \( M \): one that is 1 by \( M \), and one that is \( A \) by \( B \).

3. For each of the numbers below, determine whether it is a prime number. If it is not a prime number, factor the number into a product of prime numbers.
   a. 8303
      This isn’t prime, since \( 8303 = 19 \cdot 23 \).
   d. 80,000
      This clearly isn’t prime, since it’s divisible by 10,000. In fact \( 80,000 = 2^3 \cdot 10,000 = 2^3 \cdot 10^4 = 2^3 \cdot (2 \cdot 5)^4 = 2^7 \cdot 5^4 \).

4. Given that \( 792 = 2^3 \cdot 3^2 \cdot 11 \), find all the factors of 792 and explain your reasoning.

   A factor of 792 must have the form \( 2^i \cdot 3^j \cdot 11^k \), where \( i = 0, 1, 2, 3, j = 0, 1, 2, \) and \( k = 0, 1 \). So there are 4 choices for the power of 2, 3 choices for the power of 3, and 2 choices for the power of 11, giving us \( 4 \cdot 3 \cdot 2 = 24 \) possibilities altogether.

   Here are the factors:
   \[
   \begin{align*}
   2^03^011^0 &= 1 & 2^03^111^0 &= 3 & 2^03^211^0 &= 9 \\
   2^13^011^0 &= 2 & 2^13^111^0 &= 6 & 2^13^211^0 &= 18 \\
   2^23^011^0 &= 4 & 2^23^111^0 &= 12 & 2^23^211^0 &= 36 \\
   2^33^011^0 &= 8 & 2^33^111^0 &= 24 & 2^33^211^0 &= 72 \\
   2^03^011^1 &= 11 & 2^03^111^1 &= 33 & 2^03^211^1 &= 99 \\
   2^13^011^1 &= 22 & 2^13^111^1 &= 66 & 2^13^211^1 &= 198 \\
   2^23^011^1 &= 44 & 2^23^111^1 &= 132 & 2^23^211^1 &= 396 \\
   2^33^011^1 &= 88 & 2^33^111^1 &= 264 & 2^33^211^1 &= 792 
   \end{align*}
   \]

5. Without calculating the number \( 19 \times 23 + 1 \), explain why this number is not divisible by 19 or 23.
This number is not divisible by 19. If it were, you could make it with equal groups having 19 objects in each group. But you already have 23 groups of 19, with one object left over. You can’t make another group of 19 out of the one object remaining. (Another way of saying this is \((19 \times 23 + 1) \div 19 = 23\), remainder 1.

A similar argument holds for 23. If you divide this number of objects into groups of 23, you get 19 groups with 1 object left over.

**Section 8.5**

1. Why do we not talk about a *greatest* common multiple and a *least* common factor?

There are infinitely many common multiples, since any multiple of the least common multiple is still a common multiple. So there isn’t any greatest common multiple.

There is a least common factor, but it’s always 1 (for any two counting numbers). So this isn’t interesting or useful.

3. Show how to use the definition of GCF to determine the greatest common factor of 30 and 77.

The factors of 77 are 1, 7, 11, and 77. Since 7 and 11 aren’t factors of 30, the greatest common factor of 77 and 30 is 1.

4. Show how to use the definition of LCM to determine the least common multiple of 44 and 55.

Looking at multiples of 44, we get 44, 88, 132, 176, 220, 264, \ldots. For multiples of 55, we get 55, 110, 165, 220, 275, \ldots.

So the least common multiple of 44 and 255 is 220.

7. Show how to use the slide method to determine the GCF and LCM of 144 and 2240.

We can divide both 144 and 2,240 by 8, getting 18 and 280. And we can divide both 18 and 280 by 2, getting 9 and 140. But the only factors of 9 are 1, 3, and 9, and of these only 1 divides 140. (Take the sum of the digits of 140 to see 3 doesn’t divide it.) So we have the following table:

\[
\begin{array}{c|cc}
8 & 144, & 2,240 \\
2 & 18, & 280 \\
9 & 140 & \\
\end{array}
\]

Now we can read off the GCF as \(8 \cdot 2 = 16\) and the LCM as \(8 \cdot 2 \cdot 9 \cdot 140 = 20,160\).

9. Find the GCF and LCM of \(2^5 \cdot 3^2 \cdot 5\) and \(2^3 \cdot 3^4 \cdot 7\) without calculating the products. Explain your reasoning.
Both the GCF and LCM of \(2^5 \cdot 3^2 \cdot 5\) and \(2^3 \cdot 3^4 \cdot 7\) are product of a power of 2, a power of 3, a power of 5, and a power of 7. (The powers could be 0.)

Consider the GCF first. The power of 2 can’t be bigger than 3, since if it were, the GCF couldn’t be a factor of \(2^3 \cdot 3^4 \cdot 7\). And the power of 3 can’t be bigger than 2, since if it were, the GCF couldn’t be a factor of \(2^5 \cdot 3^2 \cdot 5\). Since 5 and 7 aren’t common factors of the two numbers, they can’t be factors of the GCF. So the GCF is \(2^3 \cdot 3^2\).

For the LCM, all the factors in the prime factorization of each number must occur in the LCM. So the power of 2 has to be at least 5, the power of 3 has to be at least 3, and the powers of 5 and 7 must be at least 1. That makes the LCM \(2^5 \cdot 3^3 \cdot 5 \cdot 7\).

11. Describe in general how to find the GCF and LCM of two counting numbers from the prime factorizations of the two numbers, and explain why the method works.

The prime factorization of the GCF of two counting numbers is the product of the smaller power of the common prime factors occurring in the two numbers. The prime factorization of the LCM is the product of the larger power of each prime factor occurring in the two numbers. (If we write the two numbers as products of powers of the same primes, using an exponent of 0 when a prime is not a factor of one of the numbers, we can say “each prime factor” for both cases, though we’re stretching things a bit by considering a prime to the 0 power as a “prime factor”.)

16. Suppose you are teaching students about least common multiples and you use only the following examples to illustrate the concept: 4 and 7; 3 and 10; 5 and 12; 8 and 9. What misconceptions about the LCM might the students develop? Explain why. What other kinds of examples should the students see? Describe some examples that you think are good, and explain why you think they are good.

From these examples, students are likely to develop the misconception that the LCM of two numbers is simply the product of the two numbers. (Note that in all these examples, the GCF is 1.) You might also include some examples, such as 5 and 15, where the LCM is one of the numbers. And you should include some examples, such as 24 and 36, where the LCM is neither the product nor one of the given numbers.

17. Kwan and Clevere are playing drums together, making a steady beat. Kwan beats the cymbals on beats that are multiples of 8. Clever beats the cymbals on beats that are multiples of 12. Find the first 4 beats on
which both Kwan and Clevere will beat the cymbals. Use mathematical terms to describe the first beat, and all the beats, on which Kwan and Clevere beat the cymbals. Explain.

They will both beat the cymbals on beats that are common multiples of 8 and 12. The LCM of 8 and 12 is 24, so that’s the first beat where they both hit the cymbals, and they will both hit the cymbals on every multiple of 24.

20. In a clothing factory, a worker can sew 18 Garment A seams in a minute and 30 Garment B seams in a minute. If the factory manager wants to complete equal numbers of Garments A and B every minute, how many workers should she hire for each type of garment? Give three different possibilities, and find the smallest number of workers the manager could hire. Explain your answers.

(Assume that both garments need the same number of seams to finish the garment, or that we really just need the same number of seams of each type per minute.) The number of seams sewn per minute for Garment A is 18 times the number of workers doing type A garments, and the number of seams sewn per minute for Garment B is 30 times the number of workers sewing type B garments.

For these to be the same, the number of seams of each type must be a common multiple of 18 and 30. Since $18 = 2 \cdot 3^2$ and $30 = 2 \cdot 3 \cdot 5$, the LCM of 18 and 30 is $2^2 \cdot 3^2 \cdot 5 = 90$. To get 90 seams of each type per minute, the manager would need 5 workers for type A and 3 workers for type B.

To keep the number of seams of the two types equal, the number must be a multiple of 90. To get $N \cdot 90$ seams of each type per minute, the manager would need $5 \cdot N$ workers for type A and $3 \cdot N$ workers for type B. So, for example, she could get 180 seams per minute with 10 workers for type A and 6 workers for type B, or 270 seams per minute with 15 workers for type A and 9 workers for type B.