Section 7.6

5. The price of play equipment for the school has just been reduced by 25%. The new, reduced price of the play equipment is $1500. Bob says he can find the original price (before the reduction) in the following way:

First I noticed that 25% is 1/4. Then I found 1/4 of $1500, which is $375. Next I added $1500 and $375, so the original price was $1875.

Is Bob’s method correct or not? If it’s correct say so and explain how to solve the problem in another way. If it’s not correct, explain briefly why not and show how to modify Bob’s method to solve the problem correctly.

Bob’s method is not correct. He has found 25% of the new price, while the discount was 25% of the original price. He could have solved the problem by noticing that, after the reduction, the new price is 75%, or \( \frac{3}{4} \), of the original price. That means that the discount is \( \frac{1}{3} \) of the reduced price; the original price is \( \frac{1500}{\frac{3}{4}} = 2000 \).

7. If sales taxes are 7%, then how much should you charge for an item if you want the total cost, including tax, to be $15? Explain the reasoning behind your method of calculation.

After adding the 7% tax, the amount paid is 107% of the price. So we want a price, \( p \), such that \( 1.07p = 15 \). So \( p = \frac{15}{1.07} = 14.02 \).

23. Frank’s Jewelers runs the following advertisement: “Come to our 40%-off sale on Saturday. We’re not like the competition, who raise prices by 30% and then have a 70%-off sale.”

(a) If two items start off with the same price, which gives you the lower price in the end: taking off 40%, or raising the price by 30% and then taking off 70% of the raised price?

Taking 40% off the price leaves 60% and taking 70% off leaves 30%. But if \( p \) is the original price, raising the price by 30% gives \( 1.3p \) and \( .6p > .3(1.3p) = .39p \) So you get a better price when the seller raises the price by 30% and then takes off 70% of the raised price.
(b) Consider the same problem more generally, with other numbers. For example, if you raise prices by 20% and then take off 50% (of the raised price), how does that compare with taking 30% off the original price? If you raise prices by 30% and then lower the raised price by 30%, how does that compare with the original price? Try at least two other pairs of percentages by which to raise and then lower a price. Describe what you observe.

Predict what happens in general. If you raise a price by $A\%$ and then take $B\%$ off the raised price, does that have the same result as if you had lowered the original price by $(B - A)\%$? If not, which produces the lower final price?

I’ll just do the general case. Lowering a price $p$ by $(B - A)\%$ leaves $\left(1 - \frac{(B - A)}{100}\right)p$ as the reduced price. Raising the price by $A\%$ and then taking $B\%$ off the reduced price gives $\left(1 - \frac{100 - B}{100}\right)\left(1 + \frac{A}{100}\right)p$. But

$$\left(1 - \frac{B - A}{100}\right)p = \left(1 - \frac{B}{100} + \frac{A}{100}\right)p$$

and

$$\left(1 - \frac{100 - B}{100}\right)\left(1 + \frac{A}{100}\right)p = \left(1 - \frac{B}{100} + \frac{A}{100} - \frac{AB}{100(100)}\right)p.$$

The second amount is smaller.

(c) Use the distributive property or FOIL [except you shouldn’t use FOIL!] to explain the pattern you discovered in part (b). Remember that to raise a price by 15%, for example, you multiply the price by $1 + 0.15$, whereas to lower a price by 15%, you multiply the price by $1 - 0.15$.

This is already explained above, in part (c).

Section 8.1

1. Johnny says that 3 is a multiple of 6 because you can arrange 3 cookies into 6 groups by putting $\frac{1}{2}$ of a cookie in each group. Discuss Johnny’s idea in detail: In what way does he have the right idea about what the term *multiple* means, and in what way does he need to modify his idea?

Johnny has the right idea in thinking about multiplication, but he has missed the fact that, when we talk about multiples and factors, we are restricting ourselves to counting (natural) numbers. So we’re not allowing ourselves to break the cookies in half.

6. Solve problems (a) and (b) and determine whether the answers are different or not. Explain why or why not.
(a) Josh has 1159 bottle caps in his collection. In how many different ways can Josh arrange his bottle-cap collection into groups so that the same number of bottle caps are in each group and so that there are not bottle caps left over (i.e., not in a group)?

Since $1159 = 19 \times 61$, he can make 19 groups with 61 caps in each group or 61 groups with 19 caps in each group, as well as one group with 1159 caps or 1159 groups of 1 cap in each group.

(b) How many different rectangles can be made whose side lengths, in centimeters, are counting numbers and whose area is 1159 square centimeters?

Only two rectangles are possible, one with dimensions 1 cm by 1159 cm and one with dimensions 19 cm by 61 cm. In this situation, we usually consider the rectangle that’s 19 cm wide and 61 cm long to be the same as the one that’s 61 cm wide and 19 cm long, but we might want to distinguish them. If we don’t distinguish them, we’ll get twice as many answers to part (a) as part (b). If we do consider those rectangles to be different, then the answers to part (a) and (b) are the same.

8. If $A$, $B$, and $C$ are counting numbers and both $A$ and $B$ are multiples of $C$, what can you say about $A + B$? Explain why your answer is always true, and give some examples to illustrate. Which property of arithmetic is relevant to this problem?

You can say that $A + B$ is a multiple of $C$. The reason is that if $A$ and $B$ are multiples of $C$, there are (counting) numbers $m$ and $n$ such that $A = mC$ and $B = nC$. So $A + B = mC + nC = (m + n)C$. That means that $A + B$ is a multiple of $C$. We have used the distributive property of arithmetic here. I’ll let you generate some examples.

Section 8.2

4. (a) Without determining the number of dots in the design in Figure 8.6, determine whether the number is even or odd. Explain.

The number is odd because we can pair up the darkly shaded arms with each other (pairing up the dots in those arms) and the lightly shaded arms with each other, except for the center dot. So all the dots are paired except the one in the center.

(b) Give some examples of things where you can tell right away that their number is even or odd without actually counting them.
Some examples might be eggs in a carton, classroom desks in a rectangular array where we can pair rows (and, if there is an odd number of rows, just count the desks in the last row).

9. Suppose that the difference between two counting numbers is odd. What can you say about the sum of the numbers? Explain why your answer is always correct.

The sum is odd. The difference can be seen as the amount left after pairing the smaller number with part of the larger number. The sum of the numbers will be twice the part they have in common (the size of the smaller number) plus the difference. So the difference determines whether the sum is odd or even. Since the difference in this case is odd, the sum is odd.

Section 8.3

3. According to the divisibility test for 5, to determine whether a counting number is divisible by 5, you have to check only its ones digit. If the ones digit is 0 or 5, then the number is divisible by 5. Otherwise, it is not. Give a clear and complete explanation for why this divisibility test is a valid way to determine whether a number is divisible by 5, using your own words.

Suppose you represent the number as bundled toothpicks in our usual base 10 way. If you divide the number into groups of 5, then each bundle of 10 forms two groups of 5, each bundle of 100 forms twenty groups of 5, etc. If there are no toothpicks in the ones “place” (loose, unbundled toothpicks), or exactly 5 toothpicks in the “ones place”, there will be no leftover toothpicks when we divide them into groups of 5. Otherwise, there will be toothpicks left over after making groups of 5. So dividing the toothpicks into groups of 5 results in no leftovers only when there are no loose toothpicks or when there are 5 loose toothpicks, or when the ones digit of the number is 0 or 5.

4. Beth knows the divisibility test for 3. Beth says that she can tell just by looking, and without doing any calculations at all, that the number 999,888,777,666,555,444,333,222,111 is divisible by 3. How can Beth do that? Explain why it’s not just a lucky guess.

Since each digit appears 3 times, the sum of the digits will be divisible by 3. The sum of the digits is 3(9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1).

10. (a) Find a divisibility test for 25; in other words, find a way to determine if a counting number is divisible by 25 without actually dividing the number by 25.
If the last two digits of the number form a 2-digit number that's divisible by 25 (i.e., 00, 25, 50, 75), then the original number is divisible by 25.

(b) Explain why your divisibility test for 25 is a valid way to determine whether a counting number is divisible by 25.

Given some number \(N\), let \(M\) be the same number with the last two digits replaced by “00” and let \(xy\) be the last two digits of \(N\). So \(N = M + xy\). Since \(M\) is some number of hundreds, it’s divisible by 25. Then, if \(xy\) is divisible by 25, so is \(N\).

12. (a) Find a divisibility test for 8. In other words, find a way to determine if a counting number is divisible by 8 without actually dividing the number by 8.

If the last three digits form a number divisible by 8, then the original number is divisible by 8.

(b) Explain why your divisibility test for 8 is a valid way to determine whether a counting number is divisible by 8.

If we change the last three digits of the number to 0s, the result is a collection of thousands. Since 1000 = 125 \(\times\) 8, any number of thousands is divisible by 8. That means the divisibility by 8 of the original number is the same as that of the number formed by the last 3 digits. (Note, however, that there are a lot of 3-digit numbers that are divisible by 8, so this test isn’t all that useful.)

14. Investigate the questions in the following parts (a) and (b) by considering a number of examples.

(a) If a whole number is divisible both by 6 and by 2, is it necessarily divisible by 12?

No. Consider 18 = 3 \(\times\) 6 = 9 \(\times\) 2.

(b) If a whole number is divisible by both 3 and by 4, is it necessarily divisible by 12?

Here are some examples. 12, 24, 36, 48. (It is true that any number divisible by 3 and 4 is divisible by 12. Can you explain why?)

(c) Based on your examples, what do you think the answers to the questions in (a) and (b) should be?

(d) “No” and “Yes”.

(e) Based on your answer to part (c) determine whether 3,321,297,402,348,516 is divisible by 12 without using a calculator or doing long division. Explain your reasoning.
This number is divisible by 4, since the last two digits give 16, which is divisible by 4. And the sum of the digits of this number is 60, which is divisible by 3. So our divisibility test for 3 says this number is divisible by 3. If the answer to (b) is “Yes”, then this number must be divisible by 12.