2. A company mixes different amounts of grape and peach juice, but always in the ratio 3 to 5.

(a) Explain how to reason with the value of the ratio to determine how much peach juice the company should mix with the following amounts of grape juice: 100 liters, 140 liters, \(G\) liters.

The appropriate unit rate to use is \(\frac{5}{3}\) of a liter of peach juice per liter of grape juice. This allows liters of grape juice to be our "equal groups" and then each of those groups has to have a corresponding \(\frac{5}{3}\) liter of peach juice to mix with it. So for 100 liters of grape juice we need \(100 \cdot \frac{5}{3} = \frac{500}{3} = 166\frac{2}{3}\) liters of peach juice; for 140 liters of grape juice we need \(\frac{700}{3} = 233\frac{1}{3}\) liters of peach juice; and for \(G\) liters of grape juice we need \(G \cdot \frac{5}{3}\) liters of peach juice. This uses a unit-rate approach from a multiple batches perspective.

(b) Explain how to reason with a value of the ratio in another way to determine how much peach juice the company should mix with the amounts of grape juice in part (a).

We could also view the ratio from the variable-parts perspective. To do that we recognize that if the quantity of grape juice used represents one group, the peach juice needed must represent \(\frac{5}{3}\) of a group. So if 100 liters of grape juice is one group, we need \(\frac{5}{3}\) of 100 liters of peach juice. Similarly for the other amounts.

(c) Explain how to reason with a value of the ratio to determine how much grape juice the company should mix with the following amounts of peach juice: 72 liters, 84 liters, \(P\) liters.

Here the appropriate unit rate to use is \(\frac{3}{5}\) of a liter of grape juice per liter of peach juice. If we use a multiple-batches perspective, for each liter of peach juice (for one batch), we need \(\frac{3}{5}\) liters of grape juice to mix with it. So for 72 liters of peach juice, we need \(72 \cdot \frac{3}{5} = \frac{216}{5} = 43\frac{1}{5}\) liters of grape juice. For 84 liters of peach juice, we need \(50\frac{3}{5}\) liters of grape juice, and for \(P\) liters of peach juice, we need \(P \cdot \frac{3}{5}\) liters of grape juice.

(d) Explain how to reason with a value of the ratio in another way to determine how much grape juice the company should mix with the amounts of peach juice in part (c).
From the variable-parts perspective, we see that if the quantity of peach juice represents one group, the quantity of grape juice must represent $\frac{3}{5}$ of a group. So we can use $\frac{3}{5}$ as a multiplier representing the number of groups. Of course, we get the same answers as in part (c).

3. (a) Which of the following two mixtures will have a stronger lime flavor?
   - 2 cups lime juice concentrate mixed in 5 cups water
   - 4 cups lime juice concentrate mixed in 7 cups water

   Solve this problem in two different ways: with unit rates and another way. Explain your reasoning in each case.

   Using unit rates, we see that $\frac{2}{7}$ of a cup of lime juice concentrate is in each cup of the drink using mix 1, while $\frac{4}{11}$ of a cup of lime juice concentrate is in each cup of mix 2. Since $\frac{2}{7} < \frac{4}{11}$, the lime flavor of mix 2 is stronger.

   Using multipliers, you could think of the quantity of mixture made as the group. For mix 1, the amount of lime concentrate needed is $\frac{2}{7}$ of the entire mixture. For mix 2, the amount of lime needed is $\frac{4}{11}$ of the entire mixture. So we see this way that the lime flavor in mix 2 is stronger.

   (b) A student might say the second mixture has a stronger flavor than the first mixture because the numbers for the second mixture are greater (in other words, $4 > 2$ and $7 > 5$). Even if the conclusion is correct, is the student’s reasoning valid? Explain why or why not.

   The student’s reasoning is not valid. For an extreme example, a third mixture has 5 cups of lime juice concentrate and 1000 cups of water. This mixture has larger numbers than both mixtures, but very little lime juice in each cup of water, so it would taste very weak.

5. A dough recipe calls for 3 cups of flour and $1 \frac{1}{4}$ cups of water. You want to use the same ratio of flour to make a dough with 10 cups of flour. How much water should you use?

   (a) Solve this problem by setting up an equation in which you set two fractions equal to each other.

   If $x$ is the amount of water needed for 10 cups of flour, we have
   
   \[ \frac{1 \frac{1}{4}}{3} = \frac{x}{10} \]
   
   so $x = \frac{(\frac{5}{4})10}{3} = \frac{25}{6}$. 

   (b) Interpret the two fractions that you set equal to each other in part (a) in terms of the recipe. Explain why it makes sense to set these two fractions equal to each other.
The fraction on the left represents the amount of water per 1 cup of flour in the original recipe. The fraction on the right represents the amount of water per 10 cups of flour. Since we want to use the same ratio of flour to water in both cases, these fractions are equal.

(c) Why does it make sense to cross-multiply the two fractions in part (a)? What is the logic behind the procedure of cross-multiplying?

Cross-multiplying is finding the numerators when we replace the fractions by equivalent fractions with a common denominator equal to the product of the original denominators. For fractions with a common denominator to be equal, their numerators must be equal. So the original fractions are equal exactly when the quantities we get by cross-multiplying are equal. But the equation we get after cross-multiplying is easier to solve.

(d) Now solve the problem of how much water to use for 10 cups of flour in a different way, by using the most elementary reasoning you can. Explain your reasoning clearly.

10 cups of flour is $3\frac{1}{3}$ times the amount of flour in the basic recipe, so you need $3\frac{1}{3}$ times the amount of water:

$$(1\frac{1}{4})(3\frac{1}{3}) = \left(\frac{5}{4}\right)\left(\frac{10}{3}\right) = \frac{25}{6}$$

7. Explain how to reason with unit rates to solve the following problems:

(a) Suppose you drive 4500 miles every half year in your car. At the end of $3\frac{3}{4}$ years, how many miles will you have driven?

4500 miles in a half year means there are 9000 miles in a whole year. So $3\frac{3}{4}$ groups of 9000 miles is $(3\frac{3}{4})(9000) = 33,750$ miles.

(b) Mo uses 128 ounces of liquid laundry detergent every $6\frac{1}{2}$ weeks. How much detergent will Mo use in a year?

Mo uses $128 \div 6\frac{1}{2} = 19\frac{9}{13}$ ounces per week. A (non-leap) year is $52\frac{1}{2}$ weeks. So we find $52\frac{1}{2}$ groups of $19\frac{9}{13}$ ounces. This is $(52\frac{1}{2})(19\frac{9}{13}) = 1026\frac{74}{13}$. (It’s ok if you just used 52 weeks for the year and got 1024 for the answer.)

(c) Suppose you have a 32-ounce bottle of weed killer concentrate. The directions say to mix $2\frac{1}{4}$ ounces of weed killer concentrate with enough water to make a gallon. How many gallons of weed killer will you be able to make from this bottle?

Each gallon of weed killer requires $2\frac{1}{4}$ ounces of concentrate, so we need to find the number of groups there are in 32 ounces when each group is $2\frac{1}{4}$ ounces. This is $32 \div 2\frac{1}{4} = 12\frac{3}{5}$ groups. So you will be able to make $12\frac{3}{5}$ gallons of weed killer.
11. A standard bathtub is approximately $4\frac{1}{2}$ feet long, 2 feet wide, and 1 foot deep. If water comes out of a faucet at the rate of $2\frac{1}{2}$ gallons per minute, how long will it take to fill the bathtub $\frac{3}{4}$ full? Use the fact that 1 gallon of water occupies 0.134 cubic feet.

The volume of the bathtub is $(4.5)(2)(1) = 9$ cubic feet, and we want to fill $\frac{3}{4}$ of that, which is $\frac{27}{4}$ cubic feet. In gallons, this is $6.75 \div 0.134 \approx 50.373$ gallons. At $2\frac{1}{2}$ gallons per minute, it would take about $50.373 \div 2.5 \approx 20.15$ minutes to fill the tub $\frac{3}{4}$ full.

Section 7.4

3. A type of dark chocolate is made by mixing cocoa and cocoa butter in the ratio of 5 to 2. Let $C$ be an unspecified number of grams of cocoa, which can vary, and let $B$ be the corresponding number of grams of cocoa butter needed to make that type of dark chocolate. Reason about quantities and use math drawings to find and explain three different equations that relate $C$ and $B$ (and do not include any other variables).

I won’t show a drawing, but one way to do this is to use a how-many-unit-in-1-group division to see that each part represents both $C \div 5$ and $B \div 2$, so (since all parts are the same size) we have $C \div 5 = B \div 2$.

Using the variable parts perspective and thinking of 1 group as made of $C$ grams of cocoa, the grams of cocoa butter needed is $\frac{2}{5}$ of a group. So $B = \frac{2}{5}C$.

Or we could divide the grams of cocoa butter in one group, $B$, into two parts, each with $\frac{B}{2}$ grams. The cocoa in that group would be five parts, each of the same size. So $C = 5(\frac{B}{2})$.

5. As in the text, consider biodiesel and petrodiesel mixtures that are 30% biodiesel and 70% petrodiesel. Let $B$ stand for an unspecified number of liters of biodiesel, which can vary, and let $P$ stand for the corresponding number of liters of petrodiesel in such mixtures. Reason about quantities and our definition of multiplication and use a math drawing to explain the equation $\frac{1}{3} \cdot B = \frac{1}{7} \cdot P$.

This is basically the same as Practice Exercise 2, but thinking about, e.g., taking $\frac{1}{3}$ of $B$ rather than dividing $B$ by 3.

6. Peacock Purple Paint is made by mixing red paint and blue paint in the ratio 3 to 7. Let $R$ be an unspecified number of liters of red paint, which can vary, and let $B$ be the corresponding number of liters of blue paint in Peacock Purple Paint. Reason about quantities and our definition of multiplication, and use math drawings to find and explain the following types of equations.
(a) \( \cdot R = B \), where \( c \) is a suitable constant (which you should find).

We can think of the \( R \) liters of red paint as one group. Dividing that into 3 equal parts (think of the strip diagram, which I won’t draw here but is the same idea as the diesel fuel in Figure 7.24), and see that the amount of blue paint needed is \( \frac{7}{3} \) of a group because we need 7 parts of the same size as the parts we’ve divided the red paint into. So \( B \) is \( \frac{7}{3} \) of a group where each group uses \( R \) liters, giving us the equation \( \frac{7}{3}R = B \).

(b) \( k \cdot B = R \), where \( k \) is a suitable constant (which you should find).

We can reason as in (a) above, taking \( B \) liters of blue paint as one group and seeing that the amount of red paint needed is 3 parts, each equal to \( \frac{1}{7}B \). So \( \frac{3}{7}B = R \).

(c) \( \frac{1}{u} \cdot R = \frac{1}{v} \cdot B \), where \( \frac{1}{u} \) and \( \frac{1}{v} \) are suitable unit fractions.

In the reasoning above (and corresponding to math drawings like Figure 7.24), each part represents both \( \frac{1}{3}R \) and \( \frac{1}{7}B \). So we have \( \frac{1}{3}R = \frac{1}{7}B \).

7. A company mixes fertilizer and soil in a 4 to 9 ratio by weight, but the amounts of fertilizer and soil the company mixes vary. Let \( F \) be an unspecified number of pounds of fertilizer, which can vary, and let \( S \) be the corresponding number of pounds of soil the company might use. For each of the following equations and math drawings (see the text for the drawings), discuss whether or not the equation is correct. If it is, explain how to reason about quantities and use a math drawing to obtain the equation. If the equation is not correct, explain why not and discuss how a student might come up with it.

(a) \( 4S = 9F \)

The equation and the math drawing are correct. 4 equal groups with 9 parts in 1 group is the same number of parts as 9 equal groups with 4 parts in 1 group, and all parts represent the same number of pounds.

(b) \( 9S = 4F \)

The equation is incorrect. The drawing shows 9 groups each with \( S \) pounds of soil and 4 groups, each with \( F \) pounds of fertilizer. But the number of pounds of soil shown isn’t equal to the number of pounds of fertilizer, so it’s incorrect to set \( 9S \) equal to \( 4F \).

(c) \( \frac{1}{9} \cdot S = \frac{1}{4} \cdot F \)

This is correct. The drawing shows \( S \) pounds of soil divided into 9 equal groups, each weighing \( \frac{S}{9} \) pounds, and the corresponding \( F \) pounds of fertilizer divided into 4 groups,
each weighing $\frac{F}{4}$ pounds. Since fertilizer and soil are mixed in a 4 to 9 ration, each group has the same number of pounds. The equation says that the weight of one of the 9 groups of soil is the same as the weight of one of the 4 groups of fertilizer.

Section 7.5

1. Suppose that 6 people can stuff flyers into 500 envelopes in 5 minutes. Assume all people work at the same steady rate.

(a) The relationship between the number of people stuffing envelopes and the number of minutes it takes to stuff flyers into 500 envelopes is what type of relationship? How can you tell? Make a table and a graph to show the relationship and explain how to find several of the entries. Include an entry for 5 people.

This is an inversely proportional relationship. As the number of people goes up, the amount of time goes down, and increasing the number of people by a given factor multiplies the time required by 1 over that factor. The key variable here is “person-minutes”; it takes (6 people)·(5 minutes) to stuff 500 envelopes, and dividing the number of people by some number would require multiplying the number of minutes by the same number to get the 500 envelopes stuffed. (Each person stuffs $\frac{500}{6}$ envelopes in 5 minutes, or $\frac{100}{6}$ envelopes in 1 minute. If we have $n$ people, they can stuff $\frac{100n}{6}$ envelopes in 1 minute, so it will take them $500 \div \frac{100n}{6} = \frac{30}{n}$ minutes to stuff 500 envelopes. If $t$ is the amount of time it takes for $n$ people to stuff 500 envelopes, $t = \frac{30}{n}$.

A sample table would be

<table>
<thead>
<tr>
<th>People</th>
<th>6</th>
<th>3</th>
<th>12</th>
<th>24</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minutes</td>
<td>5</td>
<td>10</td>
<td>2.5</td>
<td>1.25</td>
<td>6</td>
</tr>
</tbody>
</table>

You can draw the graph of $tn = 30$.

(b) The relationship between the number of people stuffing envelopes and the number of envelopes they can stuff in 5 minutes is what type of relationship? How can you tell? Make a table and a graph to show the relationship and explain how to find several of the entries. Include an entry for 5 people.

This is a (direct) proportional relationship. As we saw before, each person can stuff $\frac{100}{6}$ envelopes in 1 minute, so $n$ people can stuff $\frac{100n}{6}$ envelopes in 1 minute and $\frac{100}{6} nt$ envelopes in $t$ minutes, or $\frac{500}{6} n$ in 5 minutes.
If we graph this relationship with the number of people on the x-axis and the number of envelopes stuffed in 5 minutes on the y-axis, we will get a straight line through the origin with slope $\frac{500}{6}$.

A sample table would be

<table>
<thead>
<tr>
<th>People</th>
<th>Envelopes</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>500</td>
</tr>
<tr>
<td>3</td>
<td>250</td>
</tr>
<tr>
<td>12</td>
<td>1000</td>
</tr>
<tr>
<td>24</td>
<td>2000</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{2500}{6} \approx 417$</td>
</tr>
</tbody>
</table>

3. If 10 workers take 8 hours to sew a store’s order of pants, then how long would 15 workers take to sew the store’s order of pants? Assume all workers work at the same steady rate and all the pants are the same.

(a) Is the proportion $\frac{8 \text{ hours}}{10 \text{ workers}} = \frac{X \text{ hours}}{15 \text{ workers}}$ appropriate for solving this sewing problem? Why or why not?

The proportion is not appropriate because the relationship between the number of workers and the number of hours required to sew the store’s order is inversely proportional, not proportional. Each worker sews a certain fraction of the order per hour, so $N$ times as many workers would do the sewing in $\frac{1}{N}$ of the time.

(b) Explain how to reason about multiplication and division of quantities to solve the sewing problem.

We can observe that it will take $8 \cdot 10 = 80$ worker-hours to sew the order. So 15 workers would need $80 \div 15 = 5\frac{1}{3}$ hours.

9. If 6 people take 3 days to dig 8 ditches, then how long would it take 4 people to dig 10 ditches. Assume that all the ditches are the same size and take equally long to dig, and that all the people work at the same steady rate. Explain how to reason about multiplication and division with quantities to solve this problem.

One way to do this is to see that it takes $18$ person-days to dig 8 ditches, so $\frac{18}{8}$ person-days to dig 1 ditch. Digging 10 ditches will require $10 \left( \frac{18}{8} \right) = \frac{45}{2}$ person-days. With 4 people digging, this would take $\left( \frac{45}{2} \right) \left( \frac{2}{4} \right) = 5\frac{5}{8}$ days.

11. Jay and Mark run a lawn-mowing service. Mark’s mower is twice as big as Jay’s, so whenever they both mow, Mark mows twice as much as Jay in a given time period. When Jay and Mark are working together, it takes them 4 hours to cut the lawn of an estate. How long would it take Mark to mow the lawn by himself? How long would it take Jay to mow the lawn by himself? Explain your answers.
In 1 hour, Mark does as much mowing as Jay can do in 2 hours. So if they work together for 4 hours, Jay has done 4 hours of mowing and Mark has done as much mowing as Jay would do in 8 hours. So it would take Jay 12 hours to mow the lawn by himself.

12. If liquid pouring at a steady rate from hose A takes 15 minutes to fill a vat, and liquid pouring at a steady rate from hose B takes 10 minutes to fill the same vat, then how long will it take for liquid pouring from both hose A and hose B to fill the vat? Explain how to reason about quantities to solve this problem.

Suppose the vat holds $V$ gallons of liquid. Hose A delivers $\frac{V}{15}$ gallons/minute and Hose B delivers $\frac{V}{10}$ gallons/minute. Combined, they deliver $\frac{V}{15} + \frac{V}{10} = \frac{5V}{30}$ gallons/minute. At this rate, it takes $\frac{30}{5V} = 6$ minutes to deliver $V$ gallons, filling the vat. (Note that we don’t need to know what $V$ actually is.)