The textbook introduces negative numbers only by saying “If \( N \) is a number, \(-N\) is also a number” and then pointing out that we can show positive and negative numbers on a number line. The purpose of these notes is to describe negative numbers in terms of (an extension of) our counting models, and show how our interpretations of the basic arithmetic operations are represented in this model. This gives children another concrete way to make sense of negative numbers and the operations on integers (beyond the number line models discussed in the text), and to reason about what properties of the operations on whole numbers still hold true for the bigger set of integers.

1. Representing the Natural Numbers and Operations on Them

Our basic model for the natural numbers is collections of objects: a number \( n \) is represented as a collection of \( n \) objects. For instance, we could represent 3 as

\[
\begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc \\
\end{array}
\]

or 4 as

\[
\begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\end{array}
\]

(Note that the arrangement of the objects doesn’t matter, just the number of them.)

In principle, we can represent any natural number this way, though in practice it’s hard to represent large numbers and for any number bigger than 5 or 6, we have to count the objects to know how many there are. We use place value representations to overcome these obstacles, but they are more abstract—for reasoning about numbers and
operations (e.g., to convince ourselves that addition is commutative and associative), we want a more basic model.

Given this representation of natural numbers, we can represent addition by combining collections of objects. So “3 + 4” would be represented as

\[
\begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc \\
\hline
\bigcirc \\
\bigcirc
\end{array} \quad \begin{array}{c}
\bigcirc \\
\bigcirc \\
\hline
\bigcirc \\
\bigcirc
\end{array} \quad \leadsto \quad \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array}
\]

where the wiggly arrow is intended to indicate that the two collections on the left have been combined into the single collection on the right.

In fact, the way that mathematicians define addition of natural numbers is really just an abstract version (described in terms of unions of sets) of this one. All the properties of addition of natural numbers can be “proved” (at least in an informal way suitable for elementary school classrooms) using this notion of combining collections as the definition of addition.

We can then define subtraction of one natural number from another one that is at least as large as the removal of the number of objects being subtracted from a collection containing the number being subtracted from. For example, “7 − 3” would be

\[
\begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc \\
\hline
\bigcirc \\
\bigcirc
\end{array} \quad \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array} \quad \leadsto \quad \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array}
\]

Again, the way that mathematicians define subtraction for natural numbers is just an abstract version of this notion of taking away, and children can use this to justify the properties of subtraction, including its relation with addition: The statement that \( a + b = c \) tells us the same thing as \( c − a = b \) or \( c − b = a \). (Make sure you see how that works in this representation.)

2. Extending the Representation to Include Negative Numbers

The problem is how to extend this representation of the numbers and the operations. In the rest of this section, we describe one of the standard ways of doing this for elementary school students. The purpose is to give them a concrete representation and definition of negative numbers and operations on integers (positive and negative) that they can use to understand the numbers and to reason about the operations. (I won’t discuss here the various ways to motivate the introduction of negative numbers—representing losses or debts or distance with direction; having solutions for subtraction problems for any natural numbers; etc. I’ll assume for now that you, and your
students, are interested in having some kind of way of doing those things and only talk about a way to do that.)

The idea is that we introduce another kind of chip—I’ll use gray here—with the special property that a chip of the first kind and a chip of the second kind cancel each other out. (This is usually described with positive and negative electrical charges: a single positive charge and a single negative charge neutralize each other, yielding a particle with zero charge. Or you can think of the positive chips as being made out of ordinary matter and the negative ones as being made of antimatter. A pair consisting of an ordinary matter particle and an antimatter particle can be created out of nothing and can spontaneously annihilate, releasing energy in an explosion! When my daughter was young, she liked the explosion version of a positive and negative chip canceling out, so that might be popular in classrooms.)

So the collection

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

is the same as the collection with no objects, and the collection

\[
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\]

represents the same number as the collection

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

does.

This adds some complication, since now we have a lot representations of each natural number \( n \) (infinitely many, in fact). For instance, all of the collections below represent the number 3:

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

(Note that it doesn’t matter where the positive and negative chips are, just that there are three more positive ones than negative ones.)

But given any collection, we can always cancel positive and negative chips (remove pairs consisting of one white and one gray chip) until the collection consists entirely of positive chips, entirely of negative chips,
or has no chips at all. We regard a collection consisting of \( n \) negative chips, for any natural number \( n \), as representing the integer \(-n\). So

\[
\begin{array}{ccc}
\text{●} \\
\text{●●}
\end{array}
\]

represents \(-3\). So all the collections of chips that are equivalent to this one, the ones having three more gray chips than white chips, also represent \(-3\), just as all the collections having three more white chips than gray ones represent the natural number 3. (For you, as adults, I only need to say this once and you can work out the details. If you use this chip model in a classroom, kids will need to practice identifying the numbers that particular collections represent and producing equivalent representations with different numbers of chips of a particular color.)

How about the operations? Well, we can use exactly the same notion of addition by combining collections. So, for instance, we see from

\[
\begin{array}{ccc}
\text{●} \\
\text{●●} \\
\text{●●} \\
\text{●●}
\end{array}
\]

that \( 3 + (-2) = 1 \), because when we combine the two collections we have three positive chips and two negative ones. The negative ones cancel two of the positive ones, so this is equivalent to a collection with a single positive chip. Again, we can use this to determine the properties of addition of integers. For instance, like addition of natural numbers, it’s commutative and associative, but unlike addition of natural numbers, addition of integers doesn’t have the property that the sum is always bigger than the numbers being added. (Note that this is something you would probably want to point out to children, and the chip model is one way to explain that to them.)

For subtraction, the idea is the same. If we want to subtract 3 from something, we need to take away three positive chips. And, in the same way, if we want to subtract \(-3\) from something, we need to take away three negative chips. The reason we couldn’t subtract a bigger natural number from a smaller one, like 3 from 2, was that we couldn’t take away the larger number of positive chips, they simply aren’t there to take away. But our new representation gives us representations of 2 that have at least 3 positive chips (and some negative ones, of course).

So, for instance, if we start with the representation

\[
\begin{array}{ccc}
\text{●} \\
\text{●●}
\end{array}
\]

for 2, we can’t remove 3 positive chips. But if we use the representation

\[
\begin{array}{ccc}
\text{●} \\
\text{●●} \\
\text{●●}
\end{array}
\]

instead, we can take away three positive chips to get
and we see that $2 - 3 = -1$. We could have started with the representation of 2 as

\[ \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\end{array} \]

and removed three positive chips to get

\[ \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc \\
\end{array} \]

which is, of course, another representation of $-1$. So if we want to subtract a positive number $n$ from a number $m$, we need to get a representation of $m$ that has at least $n$ positive chips. If the representation we start with doesn’t have that many positive chips, we can add pairs consisting of a positive chip and a negative chip until we get an equivalent representation that has at least that many positive chips.

Similarly, if we want to subtract a negative number by removing a certain number of negative chips, we need a representation that has at least that many negative chips. We can always get such a representation by adding pairs consisting of a positive chip and negative chip to get an equivalent representation that has enough negative chips for us to do the subtraction. So to compute $1 - (-4)$, can start with the standard representation for 1 and add pairs until we have 4 negative chips:

\[ \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\end{array} \]

Taking away the four negative chips, we’re left with the five positive ones and we see that $1 - (-4) = 5$.

3. Multiplication and division

For natural numbers $A$ and $B$, our definition of $A \times B$ is the number of objects we have if we assemble $A$ groups, when each group contains $B$ objects. So we can represent $2 \times 3$ as

\[ \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc \\
\end{array} \]

where we have two rows and each row is a group of 3 positive chips. Since this array has 6 positive chips and no negative ones, we see that it represents the number 6. So $2 \times 3 = 6$.

To represent $2 \times (-3)$, we just take two groups as before, but we’d have 2 groups, each with $-3$ objects. In our charge model, we interpret this as each group being a representation of $-3$. So we can think of 2 rows, each with 3 negative chips, as
In fact, as long as each group (row) is a representation of $-3$, we’ll get a representation of $2 \times (-3)$. So this picture also shows $2 \times (-3)$:

or this one:

Both of these have two representations of $-3$, so the pictures represent $2 \times (-3)$.

This charge representation of groups makes it pretty clear why the product of a positive number and a negative number will be negative. But it’s a little harder to interpret things when the number of groups is negative. For something like $(-2) \times 3$, we could just use the commutative property of multiplication to think of it as 3 groups of $-2$, but if we’re multiplying two negative numbers together, we can’t avoid a negative number of groups that way.

If we go back to our interpretation of $A \times B$ (when $A$ and $B$ are positive) as the number of objects we have if we assemble or put down $A$ groups, each with $B$ objects, it makes sense to think about $A \times B$ with $A$ negative as the number of objects we have if we start with no objects and remove $-A$ (note that $-A$ is positive!) groups, each with $B$ objects.

So for $(-2) \times 3$, we want to see how many we have if we start with no objects and take away 2 groups of 3. Since we don’t have any objects at the start, we have to pick a representation of 0 that lets us take away two groups of 3. We can use two groups of 3 positive/negative pairs:

When we remove 2 groups of $-3$, we see:
The effect of removing 2 groups of $-3$, is to leave us with 2 groups of $3$, and we see that $(-2) \times (-3) = 6$.

In general, removing groups of negative chips will leave positive chips, explaining why the product of two negative numbers should be positive.

For division, recall that we can always think of the division problem $A \div B$ as multiplication with a missing factor, $? \times B = A$ or $B \times ? = A$. Then you can conclude from what we’ve seen for multiplication that, e.g., a negative number divided by a negative number will be positive and a negative number divided by a positive number will be negative.