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2-COHOMOLOGY OF SOME UNITARY GROUPS

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In [1], we showed that the 2-cohomology of the group $SU(n, q)$ with coefficients in the standard module $V = \mathbf{F}_{q^2}^n$ is generally zero. For $SU(2, q)$, which is, of course, equal to $SL(2, q^2)$, the only exceptions occur at $q = 2^k$ with $k \geq 2$; in unpublished work, McLaughlin has shown that the second cohomology group has dimension 1 over $\mathbf{F}_{q^2}$. For $n > 2$ and $q > 3$, the only possible exceptions are at $n = 3$ with $q = 4$ or 3^k and $n = 4$ with $q = 4$. In this paper, we prove that $H^2(SU(n, q), V)$ has dimension 1 over $\mathbf{F}_{q^2}$ in the first case and vanishes in the second. We also show that $H^2(SU(3, 3), V)$ is zero.

In Section 1, we outline some basic results on the cohomology of groups. In the second section, we compute $H^2(SU(3, q), V)$ with $q = 4$ or 3^k, $k > 1$, while the 2-cohomology of $SU(4, 4)$ is determined in the third section. Finally, we show $H^2(SU(3, 3), V) = 0$ in the fourth section.

1. In this section, we describe some results on the cohomology of groups which will be needed later. For a more complete discussion, the reader is referred to [2] and [5].

Let $1 \to A \to G \to X \to 1$ be an exact sequence of groups and let V be a (left) G-module. From the Lyndon-Hochschild-Serre spectral sequence we get the exact sequence

$$H^2(X, V^A) \to H^2(G, V)_0 \to H^1(X, H^1(A, V)),$$

where V^A denotes the set of A-fixed points of V and $H^2(G, V)_0$ is the kernel of the restriction map res: $H^2(G, V) - H^2(A, V)^X$.

If A and V are finite elementary abelian p-groups and $V^A = V$, we have the exact sequence of X-modules

$$0 \to \text{Hom}\,(A, V) \xrightarrow{\mu} H^2(A, V) \xrightarrow{\varepsilon} \text{Alt}_2\,(A, V) \to 0,$$

where $\text{Alt}_2\,(A, V)$ is the group of alternating $\mathbf{F}_p$ bilinear forms from A to V. μ is the Bockstein operator with $\mu(h)$ equal to the class of the 2-cocycle

$$\mu_1(h)(a, b) = ((h(a) + h(b))^p - h(a)^p - h(b)^p)/p,$$

and ε is defined at the cocycle level by $\varepsilon(f)(a, b) = f(a, b) - f(b, a)$.

We shall be most interested in the case where A and V are vector spaces over some finite field K and $\dim_K A = 1$. In this situation we can take advantage of special direct sum decompositions of $\text{Hom}\,(A, V)$ and $\text{Alt}_2\,(A, V)$ to simplify

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the computations. We have Hom (A, V) isomorphic as X-module to $\amalg H_\sigma(A, V)$, where $H_\sigma(A, V)$ is the group of K-semilinear transformations from A to V with component automorphism σ and σ ranges over the Galois group of K over $\mathbf{F}_p$. Also, Alt_2 (A, V) is isomorphic as X-module to $\amalg A_{\sigma,\tau}(A, V)$, where

$$A_{\sigma,\tau}(A, V) = \{\phi\colon A \times A \to V \mid \phi(a, b) = T(a^\sigma \otimes b^\tau - b^\sigma \otimes a^\tau) \text{ for some } T \varepsilon \operatorname{Hom}_K(A \otimes A, V)\}$$

and σ and τ range over the Galois group of K with $\sigma < \tau$ in some fixed ordering. The second decomposition is due to Landázuri [4], who proved a related, but more complicated, result for $\dim_K A > 1$.

Suppose G is a finite group and V is KG-module for some field K of characteristic p. Multiplication by any $r \in \mathbf{Z}$ with $(r, p) = 1$ is an automorphism of $H^n(G, V)$. If $S \leq G$, we have the corestriction map cor: $H^n(S, V) \to H^n(G, V)$, and the composition

$$\text{cor} \circ \text{res}\colon H^n(G, V) \to H^n(G, V)$$

is multiplication by the index $[G:S]$. It follows that if $[G:S]$ is prime to p, cor $\circ$ res is an automorphism and res is injective. If S is normal in G, we get $H^n(G, V) \cong H^n(S, V)^{G/S}$, but for any subgroup S with $[G:S]$ prime to p, $\dim_K H^n(S, V)^{G/S} \geq \dim_K H^n(G, V)$.

Finally, we give a condition which can sometimes be used to show that a cocycle is not a coboundary. Let A be an abelian group and V a trivial A-module. Let $\hat{V}$ be the free abelian group on the non-zero elements of V. Then if $f \in Z^n(A, V)$, we define $\Delta f\colon A^n \to \hat{V}$ by

$$f(a_1, \ldots, a_n) = \sum_{\sigma \in \Sigma_n} (\operatorname{sgn} \sigma) f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

If f is a coboundary, the terms of $\Delta f(a_1, \ldots, a_n)$ cancel in pairs and we see that $\Delta f = 0$.

2. In this section, we show that $H^2(SU(3, q), V)$ has dimension 1 over $\mathbf{F}_{q^2}$ when $q = 3^k$, $k > 1$, or 4 and V is the standard module $\mathbf{F}_{q^2}^3$. We begin by establishing a lower bound for the dimension of the cohomology group when $q = 3^k$.

McLaughlin has shown that, for $j > 1$, $H^2(SL(3, 3^j), \mathbf{F}_{3^j}^3) \cong \mathbf{F}_{3^j}$. In the following lemma, we show that the restriction map $H^2(SL(3, 3^{2k}), V) \to H^2(SU(3, 3^k), V)$ is non-zero.

LEMMA 1. *Suppose* $[\tilde{f}] \in H^2(SL(3, 3^{2k}), V)$, $k > 1$, *is not the zero class. Then*

$$\text{res } [\tilde{f}] = [f] \in H^2(SU(3, 3^k), V)$$

is also non-zero.

Proof. Obviously, it is enough to consider the restriction to a 3-Sylow subgroup of $SU(3, 3^{2k})$. For convenience, let $K = \mathbf{F}_{3^{2k}}$ and let $a \mapsto \bar{a}$ be the involutory automorphism. We take

$$U = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b \in K, b + \bar{b} + a\bar{a} = 0 \right\}$$

for the 3-Sylow. Then, for elements

$$x = \begin{pmatrix} 1 & a & b \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & c & d \\ 0 & 1 & -\bar{c} \\ 0 & 0 & 1 \end{pmatrix}$$

of U we can take

$$f(x, y) = \begin{pmatrix} \sqrt[3]{ad + a^2c} \\ 0 \\ 0 \end{pmatrix}$$

We need to show this is not a coboundary.

Suppose to the contrary that $f = \delta g$. Writing

$$g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{pmatrix},$$

we must have

$$\begin{pmatrix} g_1(x) + g_1(y) + ag_2(y) + bg_3(y) - g_1(xy) \\ g_2(x) + g_2(y) - \bar{a}g_3(y) - g_2(xy) \\ g_3(x) + g_3(y) - g_3(xy) \end{pmatrix} = \begin{pmatrix} \sqrt[3]{ad + a^2c} \\ 0 \\ 0 \end{pmatrix}$$

Taking $y \in Z(U)$ so $c = 0$, we get

$$g_2(x) + g_2(y) - \bar{a}g_3(y) - g_2(xy) = g_2(y) + g_2(x) - g_2(yx).$$

Since $xy = yx$, we have $\bar{a}g_3(y) = 0$. Thus g_3 vanishes on $Z(U)$.

Keeping $y \in Z(U)$, we also have

$$g_1(x) + g_1(y) + ag_2(y) - g_1(xy) = \sqrt[3]{ad}$$

and

$$g_1(y) + g_1(x) + dg_3(x) - g_1(yx) = 0.$$

These imply $ag_2(y) - dg_3(x) = \sqrt[3]{ad}$. Fixing x with $a = 1$, we see that

$$\alpha = \frac{g_2(y) - \sqrt[3]{d}}{d}$$

is a constant for $y \in Z(U)$. Thus $a(\alpha d + \sqrt[3]{d}) - dg_3(x) = \sqrt[3]{ad}$ for all non-identity x in P, and so

$$g_3(x) = \frac{\sqrt[3]{d}(a + \sqrt[3]{a}) + a\alpha d}{d} = d^{-2/3}(a + \sqrt[3]{a}) + a\alpha.$$

Since $g_3(x)$ is independent of d, $d^{-2/3}$ must be constant for all $d \in K$ with $d + \bar{d} = 0$. It follows that $|K| \leq 9$, a contradiction.

THEOREM 1. *Let* $K = \mathbf{F}_{q^2}$ *and let* $V = K^3$ *be the standard module for* $SU(3, q)$. *If* $q = 4$ *or* 3^k, $k > 1$, *then* $\dim_K H^2(SU(3, q), V) = 1$.

Proof. We will first compute $H^2(B, V)$ for B a Borel subgroup of $SU(3, q)$. Since $|SU(3, q): B|$ is prime to the characteristic of K, we have

$$\dim_K H^2(B, V) \geq \dim_K H^2(SU(3, q), V).$$

Let $a \mapsto \bar{a}$ be the involutory automorphism of K as before and let K_0 be the subfield fixed by that automorphism.

As in the preceding lemma, let U be the Sylow subgroup

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b \in K, \quad b + \bar{b} + a\bar{a} = 0 \right\}$$

and let

$$T = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & \bar{t}/t & 0 \\ 0 & 0 & 1/\bar{t} \end{pmatrix} \middle| t \in K^\times \right\}.$$

Then B is the semidirect product TU. Let $Z = Z(U)$; Z is a normal subgroup of B and we have the exact sequences

$$0 \to H^2(B, V)_0 \to H^2(B, V) \to H^2(Z, V)^{B/Z},$$

$$H^2(B/Z, V^Z) \to H^2(B, V)_0 \to H^1(B/Z, H^1(Z, V)).$$

LEMMA 2. $H^2(Z, V)^{B/Z} = 0$.

Proof. Suppose $f \in Z^2(Z, V)^T$. Writing

$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

as usual and identifying Z with $\{b \in K \mid b + \bar{b} = 0\}$ in the obvious fashion, we have

$$\begin{pmatrix} t f_1\left(\frac{1}{t\bar{t}} b_1, \frac{1}{t\bar{t}} b_2\right) \\ \frac{\bar{t}}{t} f_2\left(\frac{1}{t\bar{t}} b_1, \frac{1}{t\bar{t}} b_2\right) \\ \frac{1}{\bar{t}} f_3\left(\frac{1}{t\bar{t}} b_1, \frac{1}{t\bar{t}} b_2\right) \end{pmatrix} = \begin{pmatrix} f_1(b_1, b_2) \\ f_2(b_1, b_2) \\ f_3(b_1, b_2) \end{pmatrix}$$

for all b_1, $b_2 \in Z$ and all $t \in K^\times$. Take $y \in K \backslash K_0$ and put $t = \bar{y}/y$. Then $t\bar{t} = 1$ but none of t, $\bar{t}/t$ and $1/\bar{t}$ is 1. This shows $f = 0$, so $Z^2(Z, V)^T = 0$. Since T is a p'-group, for p the characteristic of K, this implies $H^2(Z, V)^T = 0$ and finally $H^2(Z, V)^{B/Z} = 0$.

LEMMA 3. *If $q = 3^k$, $k > 1$, or $q = 4$, then $H^2(B/Z, V^Z) = 0$.*

Proof. Let $\{e_1, e_2, e_3\}$ be the standard basis for V, so $V^Z = \langle e_1, e_2 \rangle$. We have

$$H^2(B/Z, V^Z) = H^2(U/Z, V^Z)^T$$

since T is a p'-group. We determine $H^2(U/Z, \langle e_1 \rangle)^T$ and $H^2(U/Z, V^Z/\langle e_1 \rangle)^T$, and use this information to find $H^2(U/Z, V^Z)^T$.

We have the exact sequence

$$0 \to \operatorname{Hom}\,(U/Z, \langle e_1 \rangle)^T \to H^2(U/Z, \langle e_1 \rangle)^T \to \operatorname{Alt}_2\,(U/Z, \langle e_1 \rangle)^T \to 0.$$

We use the decompositions for Hom and Alt_2 described in the first section, making the obvious identification of U/Z with K. If $h \in H_\sigma(U/Z, \langle e_1 \rangle)^T$, $th(\bar{t}^2 a/t) = h(a)$, so $h \neq 0$ implies $t(\bar{t}^2/t)^\sigma = 1$ for all $t \in K^\times$. Then $t \mapsto t^2/\bar{t}$ must be an automorphism of K, but taking $t \neq 0, -1$, this means

$$1 + \frac{\bar{t}}{t^2} = \frac{(1+t)^2}{1+\bar{t}}.$$

This leads to $t^2 - 2t\bar{t} - \bar{t}^2 = (t - \bar{t})^2 = 0$ for all $t \in K$, which is absurd. We conclude that $\operatorname{Hom}\,(U/Z, \langle e_1 \rangle)^T = 0$.

If $\phi \neq 0$ belongs to $A_{\sigma,\tau}(U/Z, \langle e_1 \rangle)^T$, $t(\bar{t}/t^2)^\sigma(\bar{t}/t^2)^\tau = 1$ for all $t \in K^\times$. If $-1 \in K$, this is clearly false, so suppose $q = 4$ where the condition reads $tt^{2\sigma}t^{2\tau} = 1$. Writing $t^\sigma = t^{2^x}$ and $t^\tau = t^{2^y}$ with $0 \leq x < y < 4$, we obtain

$$1 + 2^{x+1} + 2^{y+1} \equiv 0 \pmod{2^4 - 1}.$$

It is easy to see that this congruence has no solutions, giving $\operatorname{Alt}_2\,(U/Z, \langle e_1 \rangle)^T = 0$. Thus $H^2(U/Z, \langle e_1 \rangle)^T = 0$.

Now we use the exact sequence to compute $H^2(U/Z, V^Z/\langle e_1 \rangle)^T$. If

$$h \in H_\sigma(U/Z, V^Z/\langle e_1 \rangle)^T,$$

$h \neq 0$ implies

$$\frac{\bar{t}}{t}\left(\frac{\bar{t}}{t^2}\right)^\sigma = 1 \quad \text{for all } t \in K^\times.$$

Taking $t \neq 0, 1$ in K_0 shows this is impossible, so $\operatorname{Hom}\,(U/Z, V^Z/\langle e_1 \rangle)^T = 0$. If $\phi \neq 0$ belongs to $A_{\sigma,\tau}(U/Z, V^Z/\langle e_1 \rangle)^T$, we must have

$$\left(\frac{\bar{t}}{t}\right)\left(\frac{\bar{t}}{t^2}\right)^\sigma\left(\frac{\bar{t}}{t^2}\right)^\tau = 1 \quad \text{for all } t \in K^\times.$$

Looking at K_0, we see this implies $q \leq 4$, so assume $q = 4$ where the condition reads $t^3 t^{2\sigma} t^{2\tau} = 1$. Solving a congruence as above, we find that we can take $t^\sigma = t^2$ and $t^\tau = t^4$, so $\phi(a_1, a_2) = (a_1^2 a_2^4 - a_2^2 a_1^4)a$ for some $a \in K$.

We obtain a cocycle mapping onto this alternating form as follows. Choose an $\mathbf{F}_2$-basis for U/Z, say $x_1, \ldots, x_n$, and define $f_2(x_i, x_j)$ to be $\phi(x_i, x_j)$ if $i < j$ and 0 otherwise. Then $f_2(a_1, a_2) - f_2(a_2, a_1) = \phi(a_1, a_2)$ for all $a_1, a_2 \in U/Z$. Suppose we can find a cocycle $f \in Z^2(U/Z, V^Z)^T$ projecting onto f_2. Then

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

and the cocycle condition gives $a_1 f_2(a_2, a_3) = \delta f_1(a_2, a_2, a_3)$. Thus, if we put $g(a_1, a_2, a_3) = a_1 f_2(a_2, a_3)$, $\Delta g = 0$. Taking $a_1 = 1$, $a_2 = \omega$, a primitive cube root of 1, and $a_3 \notin K_0$ along with a suitably chosen $\mathbf{F}_2$-basis, we see that this is not the case unless $a = 0$. Thus $f_2 = 0$ and $f_1 \in Z^2(U/Z, \langle e_1 \rangle)^T$. Combining this with the fact that $H^2(U/Z, \langle e_1 \rangle)^T = 0$ completes the proof.

LEMMA 4. *If* $q = 3^k$, $k > 1$, *or* $q = 4$, *then* $\dim_K H^1(B/Z, H^1(Z, V)) = 1$.

Proof. Again, it suffices to consider $H^1(U/Z, H^1(Z, V))^T$. Take $d \in Z^1(Z, V)$ and write

$$d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

We have

$$\begin{pmatrix} d_1(b_1 + b_2) \\ d_2(b_1 + b_2) \\ d_3(b_1 + b_2) \end{pmatrix} = \begin{pmatrix} d_1(b_1) + d_1(b_2) + b_1 d_3(b_2) \\ d_2(b_1) + d_2(b_2) \\ d_3(b_1) + d_3(b_2) \end{pmatrix}.$$

We note that d_2 and d_3 are homomorphisms of abelian groups. The coboundaries $B^1(Z, V)$ are the maps

$$b \mapsto \begin{pmatrix} bv_3 \\ 0 \\ 0 \end{pmatrix}$$

for some $v_3 \in K$, so if we write $g \in Z^1(U/Z, H^1(Z, V))^T$ as

$$g(a) = \begin{pmatrix} g_1^a \\ g_2^a \\ g_3^a \end{pmatrix},$$

we see that $a \mapsto g_3^a$ belongs to

$$H^1(U/Z, H^1(Z, V/\langle e_1, e_2 \rangle))^T = \mathrm{Hom}\,(U/Z, \mathrm{Hom}\,(Z, V/\langle e_1, e_2 \rangle))^T.$$

We show this map is zero. If $q = 4$, take $b_1 = b_2$ in the expression for $d(b_1 + b_2)$ above to see $d_3 = 0$ for all $d \in Z^1(Z, V)$. If $q = 3^k$, put $W = V/\langle e_1, e_2\rangle$. We have

$$\operatorname{Hom}(U/Z, \operatorname{Hom}(Z, W))^T = \coprod_{\tau,\sigma} H_\tau(U/Z, H_\sigma(Z, W))^T,$$

where τ runs through the Galois group of K over the prime field and σ runs through the Galois group of K_0.

If $h \in H_\tau(U/Z, H_\sigma(Z, W))^T$,

$$h(a)(b) = \frac{1}{\bar{t}}\left(\frac{\bar{t}}{t^2}\right)^\tau \left(\frac{1}{t\bar{t}}\right)^\sigma h(a)h(b) \quad \text{for all } t \in K^\times,$$

so if $h \neq 0$,

$$\frac{1}{\bar{t}}\left(\frac{\bar{t}}{t^2}\right)^\tau \left(\frac{1}{t\bar{t}}\right)^\sigma = 1 \quad \text{for all } t \in K^\times.$$

On K_0, this condition reads

$$\frac{1}{t}\left(\frac{1}{t}\right)^\tau \left(\frac{1}{t^2}\right)^\sigma = 1,$$

and, writing $t^\tau = t^{3^x}$ on K_0 and $t^\sigma = t^{3^y}$, we have

$$1 + 3^x + 2 \cdot 3^y \equiv 0 \pmod{3^k - 1} \quad \text{with } 0 \le x, y < k.$$

The only solution is $x = 0$, $y = 1$ and $k = 2$, so we have $t^\tau = t$ or $\bar{t}$, $t^\sigma = t^3$ and $\bar{t} = t^9$. Checking, we see that the original condition fails on $\mathbf{F}_{81}$, and we conclude that $H^1(U/Z, H^1(Z, W))^T = 0$. This implies that $g_3^a = 0$ for all $a \in U/Z$.

Identifying Z with K_0 as usual, we have $B^1(Z, V) \cong \operatorname{Hom}_{K_0}(Z, \langle e_1\rangle)$. Then

$$H^1(U/Z, H^1(Z, V))^T \cong H^1(U/Z, \operatorname{Hom}(Z, \langle e_1, e_2\rangle)/\operatorname{Hom}_{K_0}(Z, \langle e_1\rangle))^T,$$

since $g_3^a = 0$ implies g_1^a is a homomorphism. We write $\operatorname{Hom}(Z, \langle e_1, e_2\rangle)/\operatorname{Hom}_{K_0}(Z, \langle e_1\rangle)$ as

$$\operatorname{Hom}_{K_0}(Z, \langle e_2\rangle) \oplus \coprod H_\sigma(Z, \langle e_1, e_2\rangle),$$

where σ runs through the non-identity members of the Galois group of K_0 over $\mathbf{F}_3$.

Suppose $d \in Z^1(U/Z, H_\sigma(Z, \langle e_1, e_2\rangle))^T$ for some $\sigma \neq 1$. If $q = 4$, we have $\sigma = {}^-$, and stability gives

$$\begin{pmatrix} t\, d_1^{t^{2}a}\left(\frac{1}{t^5}b\right) \\ \frac{\bar{t}}{t} d_2^{t^{2}a}\left(\frac{1}{t^5}b\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{t^9} d_1^{t^{2}a}(b) \\ \frac{1}{t^7} d_2^{t^{2}a}(b) \end{pmatrix} = \begin{pmatrix} d_1^a(b) \\ d_2^a(b) \end{pmatrix}.$$

If $a \neq 0$, we can take $t = a^{-1/2} = a^7$ to get $a^{-3}\, d_1^1(b) = d_1^a(b)$ and $a^{-4}\, d_2^1(b) = d_2^a(b)$. The cocycle condition reads $d_2^{a_1+a_2}(b) = d_2^{a_1}(b) + d_2^{a_2}(b)$ so if $d_2 \neq 0$, we must have $a \mapsto a^{-4}$ additive. This is not the case on $\mathbf{F}_{16}$, so $d_2 = 0$. Now we get $d_1^{a_1+a_2}(b) = d_1^{a_1}(b) + d_1^{a_2}(b)$ so if $d_1 \neq 0$, $a \mapsto a^{-3}$ must be additive. This isn't true either, so $d = 0$.

Keep $q = 4$ and take $d \in Z^1(U/Z, \operatorname{Hom}_{K_0}(Z, \langle e_2 \rangle))^T$. This time stability gives

$$\frac{1}{t^2} d^{t^2 a}(b) = d^a(b).$$

and, taking $t = a^7$ again, we see $ad^1(b) = d^a(b)$. Since d^a is K_0-linear, we have $d^a(b) = abve_2$ for some $v \in K$. Thus, if $q = 4$, $\dim_K H^1(U/Z, H^1(Z, V)) = 1$ and the group is generated by $[d]$, where $d(a)$ is the cohomology class of the map

$$b \mapsto \begin{pmatrix} 0 \\ ab \\ 0 \end{pmatrix}.$$

Now assume $q = 3^k$, $k > 1$, and take $d \in Z^1(U/Z, H_\sigma(Z, \langle e_1, e_2 \rangle))^T$ for some $\sigma \neq 1$. From stability, we have

$$\frac{\bar{t}}{t} d_2^{\bar{t}a/t^2}\left(\frac{1}{t\bar{t}} b\right) = d_2^a(b).$$

Taking $t = -1$, this reads $d_2^{-a}(b) = d_2^a(b)$, but the cocycle condition gives $d_2^{a_1+a_2}(b) = d_2^{a_1}(b) + d_2^{a_2}(b)$, so

$$0 = d_2^{a_1-a_1}(b) = d_2^{a_1}(b) + d_2^{-a_1}(b) = 2d_2^{a_1}(b).$$

Thus $d_2 = 0$ and $d_1 \in \operatorname{Hom}(U/Z, H_\sigma(Z, \langle e_1 \rangle)^T$. Suppose

$$d_1 \in H_\tau(U/Z, H_\sigma(Z, \langle e_1 \rangle))^T$$

for some τ in the Galois group of K. Then we have

$$t d_1^{\bar{t}a/t^2}\left(\frac{1}{t\bar{t}} b\right) = t\left(\frac{\bar{t}}{t^2}\right)^\tau \left(\frac{1}{t\bar{t}}\right)^\sigma d_1^a(b),$$

so if $d_1 \neq 0$, $t(\bar{t}/t^2)^\tau(1/t\bar{t}) = 1$ for all $t \in K^\times$. On K_0, this reads $t = t^\tau t^{2\sigma}$, so writing $t^\tau = t^{3^x}$ on K_0 and $t^\sigma = t^{3^y}$, we have $3^x + 2 \cdot 3^y - 1 \equiv 0 \pmod{3^k - 1}$ with $0 \leq x, y < k$. The only solution is $x = z = k - 1$, so $t^\sigma = \sqrt[3]{t}$ and $t^\tau = \sqrt[3]{t}$ or $\sqrt[3]{\bar{t}}$. Returning to K, we see t^τ must be $\sqrt[3]{t}$.

This argument also shows $H^1(U/Z, \operatorname{Hom}_{K_0}(Z, \langle e_2 \rangle))^T = 0$, so

$$\dim_K H^1(U/Z, H^1(Z, V))^T = 1.$$

Lemmas 2 through 4 and the exact sequences imply that $\dim_K H^2(B, V) \leq 1$, so $\dim_K H^2(SU(3, q^2), V) \leq 1$. With Lemma 1, this completes the argument for

$q = 3^k$, $k > 1$. For $q = 4$, we exhibit a cocycle class in $H^2(B, V)_0$ mapping to $[d]$ in $H^1(B/Z, H^1(Z, V))$. For

$$x_i = \begin{pmatrix} 1 & a_i & b_i \\ 0 & 1 & -\bar{a}_i \\ 0 & 0 & 1 \end{pmatrix} \in U,$$

define $f(p_1, p_2)$ to be

$$\begin{pmatrix} a_1^6 a_2 b_0 + a_1^2 a_2^5 + a_1^3 a_2^4 + (b_1 + b_2) a_1 a_2 \\ a_1 b_2 + a_1^2 a_2^4 b_0 + a_1 a_2^5 a_0 + a_1^4 a_2^2 b_0 + a_1 a_2^5 + a_1^5 a_2 \\ a_1 a_2 \end{pmatrix},$$

where b_0 is a fixed element of K with $b_0 + b_1 = 1$. Computation shows that f is fixed under the action of T and $[f]$ maps to $[d]$, so $[f] \neq 0$ in $H^2(B, V)$. Since the 2-Sylow subgroups of $SU(3, 4)$ are trivial intersection sets, $[f]$ is a stable class in $H^2(B, V)$ and so $H(SU(3, 4), V) \neq 0$. The upper bound obtained above gives $\dim_K H^2(SU(3, 4), V) = 1$. This completes the proof.

3. In this section we prove that $H^2(SU(4, 4), V) = 0$, where, as before, V is the standard module for the special unitary group. The methods used are similar to those in the preceding section and we will use much of the same notation: K is $\mathbf{F}_{16}$, K_0 is $\mathbf{F}_4$, and so on. Additionally, for $T \in GL(n, K)$, we write $\tilde{T}$ for $(\bar{T}^t)^{-1}$.

THEOREM 2. $H^2(SU(4, 4), V) = 0$.

Proof. Take $P \leq SU(4, 4)$ to be the stabilizer of a maximal totally isotropic subspace W of V. $[SU(4, 4): P]$ is prime to 2, so it suffices to show $H^2(P, V) = 0$. Since $0 \to W \to V \to V/W \to 0$ is an exact sequence of P-modules we can prove $H^2(P, V) = 0$ by showing that $H^2(P, W)$ and $H^2(P, V/W)$ both vanish. We show $H^2(P, W) = 0$ below; the arguments for $H^2(P, V/W)$ are similar.

P is a semidirect product LU where, with respect to an appropriate basis,

$$L = \left\{ \begin{pmatrix} T & 0 \\ 0 & \tilde{T} \end{pmatrix} \mid T \in GL(2, K), \det T \in K_0 \right\}$$

and

$$U = \left\{ \begin{pmatrix} 1 & H \\ 0 & 1 \end{pmatrix} \mid H \in M_2(K), \bar{H}^t = H \right\}.$$

U acts trivially on W, so we have the exact sequences

$$0 \to H^2(P, W)_0 \to H^2(P, W) \to H^2(U, W)^L,$$

$$H^2(L, W) \to H^2(P, W)_0 \to H^1(L, H^1(U, W)).$$

$H^2(L, W)$ is zero because L has central elements acting fixed-point-free on W.

LEMMA 5. $H^2(U, W)^L = 0$.

Proof. It suffices to show that Hom $(U, W)^L$ and $\text{Alt}_2(U, W)^L$ are zero. We identify U with the set of 2×2 Hermitian matrices over K and L with the set of $T \in GL(2, K)$ having determinants in K_0. With these identifications, L acts on U by $T \circ H = TH\bar{T}^t$, and it is not hard to see that U is a simple L-module.

Suppose $h \in$ Hom $(U, W)^L$. Then $Th(T^{-1}H\tilde{T}) = h(H)$ for all $H \in U$, $T \in L$, and the kernel of h is an L-submodule. Thus, if we can show ker $h \neq 0$, we have $h = 0$.

Take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix}$$

for some $t \in K$ of norm 1, $t \neq 1$. Then $T^{-1}A\tilde{T} = A$ and we have $h(A) = Th(A)$. Since T is free of eigenvalue 1, it follows that $h(A) = 0$, whence $h = 0$. Thus Hom $(U, W)^L = 0$.

Suppose $\phi \in \text{Alt}_2(U, W)^L$ with $\phi(H, U) = 0$ for some H. Since U is a simple L-module, this would imply $\phi = 0$. Keep A as above and put

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & \theta \\ \bar{\theta} & 0 \end{pmatrix}$$

where θ is a solution of $X^2 + X + \omega = 0$ in K, ω a primitive cube root of 1. $\{A, B, C, D\}$ is a K_0-basis for U and we have $\bar{\theta} = \theta^2 + \omega^2 = \theta + 1$ and $\theta\bar{\theta} = \theta + \omega + \theta = \omega$.

Let

$$R = \begin{pmatrix} a & b \\ 0 & \dfrac{1}{a} \end{pmatrix}.$$

Then

$$(1) \quad R \circ A = \begin{pmatrix} a\bar{a} & 0 \\ 0 & 0 \end{pmatrix} \qquad (2) \quad R \circ B = \begin{pmatrix} b\bar{b} & \dfrac{b}{\bar{a}} \\ \dfrac{\bar{b}}{a} & \dfrac{1}{a\bar{a}} \end{pmatrix}$$

$$(3) \quad R \circ C = \begin{pmatrix} \bar{a}b + a\bar{b} & \dfrac{a}{\bar{a}} \\ \dfrac{\bar{a}}{a} & 0 \end{pmatrix} \qquad (4) \quad R \circ D = \begin{pmatrix} \bar{a}b\bar{\theta} + a\bar{b}\theta & \dfrac{a}{\bar{a}}\theta \\ \dfrac{\bar{a}}{a}\bar{\theta} & 0 \end{pmatrix}$$

Take $\phi \in \text{Alt}_2(U, W)^L$, so $T\phi(T^{-1}H_1\tilde{T}, T^{-1}H_2\tilde{T}) = \phi(H_1, H_2)$ for all $T \in L$, $H_1, H_2 \in U$. From (1), with $a\bar{a} = 1$ and $b = 0$, we get $\phi(A, kA) = 0$ for all $k \in K_0$. From (1) and (2), again taking $a\bar{a} = 1$ and $b = 0$, we get $\phi(A,$

$kB) = 0$ for all $k \in K_0$. Choosing b so that $a\bar{b} \in K_0$, we see from (1) and (3) that

$$\phi\left(a\bar{a}A, \frac{a}{\bar{a}}C\right) = \begin{pmatrix} a & b \\ 0 & \dfrac{1}{a} \end{pmatrix}\phi(A, C).$$

Since the left side is independent of the particular choice of b, this means that

$$\phi(A, C) = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad \text{for some } x \in K.$$

Similarly,

$$\phi(A, D) = \begin{pmatrix} y \\ 0 \end{pmatrix} \quad \text{for some } y \in K.$$

We also note that, for $k \in K_0$,

$$\phi(k^2A, D) = \begin{pmatrix} ky \\ 0 \end{pmatrix}.$$

Let $\eta = \sqrt{\theta}$. We have

$$\begin{pmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{pmatrix} \circ A = \omega^2 A \quad \text{and} \quad \begin{pmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{pmatrix} \circ C = D,$$

so

$$\phi(\omega^2 A, D) = \begin{pmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{pmatrix}\phi(A, C).$$

Thus $y = \omega^2\eta x$. Also,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ A = B, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ C = C \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ D = \begin{pmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{pmatrix} = C + D,$$

so

$$\phi(B, C) = \begin{pmatrix} 0 \\ x \end{pmatrix}, \quad \phi(B, D) = \begin{pmatrix} 0 \\ x + y \end{pmatrix} \quad \text{and} \quad \phi(k^2B, D) = \begin{pmatrix} 0 \\ k(x + y) \end{pmatrix}.$$

Finally,

$$\begin{pmatrix} 0 & \eta \\ \bar{\eta} & 0 \end{pmatrix} \circ A = \omega^2 B \quad \text{and} \quad \begin{pmatrix} 0 & \eta \\ \bar{\eta} & 0 \end{pmatrix} \circ C = D,$$

so we have

$$\phi(\omega^2 B, D) = \begin{pmatrix} 0 & \eta \\ \bar{\eta} & 0 \end{pmatrix}\phi(A, C)$$

which says $\omega(x + y) = \bar{\eta}x$.

Since $y = \omega^2\eta x$, this gives $\omega x + \eta x = \bar{\eta} x$ which leads to $(\omega^2 + \theta + \bar{\theta})x^2 = 0$. As $\theta + \bar{\theta} = 1$, this implies $x = 0$, and then $y = 0$. Using the facts that

$$\begin{pmatrix}1 & 0\\ 0 & k\end{pmatrix} \circ A = A, \quad \begin{pmatrix}1 & 0\\ 0 & k\end{pmatrix} \circ C = kC \quad \text{and} \quad \begin{pmatrix}1 & 0\\ 0 & k\end{pmatrix} \circ D = kD \quad \text{for} \quad k \in K_0,$$

we have $\phi(A, kC) = \phi(A, kD) = 0$ for all $k \in K_0$.

Thus $\phi(A, U) = 0$ and we have shown that $\mathrm{Alt}_2\,(U, W)^L = 0$. This completes the proof of the lemma.

LEMMA 6. $H^1(L, H^1(U, W)) = 0$.

Proof. Since U acts trivially on W, this is really

$$H^1(L, \mathrm{Hom}\,(U, W)) \cong H^1(L, \mathrm{Hom}_{K_0}\,(U, W)) \oplus H^1(L, H_{\surd}(U, W)).$$

Suppose $h \in \mathrm{Hom}_{K_0}\,(U, W)$ and take $\lambda \in K_0$, so

$$\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} \in Z(L).$$

We have

$$\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} \circ h\right)(H) = \lambda h(\lambda^{-2}H) = \lambda^{-1}h(H),$$

so L has central elements acting fixed-point-free on $\mathrm{Hom}_{K_0}\,(U, W)$ and

$$H^1(L, \mathrm{Hom}_{K_0}\,(U, W)) = 0.$$

Thus, $H^1(L, H^1(U, W)) = H^1(L, H_{\surd}(U, W))$.

Take $f \in Z^1(X, H_{\surd}(U, W))$. Since the diagonal subgroup of L has order prime to 2, we can assume that f vanishes on the diagonal subgroup. Put

$$f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = h_x.$$

From

$$\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & x+y\\ 0 & 1\end{pmatrix},$$

we get

$$h_{x+y} = h_x + \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \circ h_y, \tag{5}$$

and from

$$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \mu^{-1}\end{pmatrix} = \begin{pmatrix}1 & \lambda\mu^{-1}x\\ 0 & 1\end{pmatrix},$$

we have

$$h_{\lambda\mu^{-1}x} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \circ h_x. \tag{6}$$

Put $h_x(A) = \binom{a(x)}{b(x)}$, where $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ as before. (5) gives

$$\begin{pmatrix} a(x+y) \\ b(x+y) \end{pmatrix} = \begin{pmatrix} a(x) + a(y) + xb(y) \\ b(x) + b(y) \end{pmatrix}.$$

Taking $x = y$, we see $b \equiv 0$ and a is additive. From (6), we have

$$\frac{\lambda}{\sqrt{\lambda\bar{\lambda}}} a(x) = a(\lambda\mu^{-1}x),$$

so taking $\lambda = 1$ and $\mu \in K_0$, we see $a(x) = a(\mu^{-1}x)$ for all $\mu \in K_0$. The additivity of a then implies $a(x) = 0$.

Next put $h_x(C) = \binom{c(x)}{d(x)}$. Again we see from (5) that $d = 0$ and c is additive. From (6), we have

$$\frac{\lambda}{\sqrt{\lambda\mu}} c(x) = c(\lambda\mu^{-1}x),$$

so for $\lambda \in K_0$, $\sqrt{\lambda}\, c(x) = c(\lambda x)$. Then $c(x) = \sqrt{x}\, c(1)$ for $x \in K_0$. Now let $h_x(D) = \binom{r(x)}{s(x)}$. Again we have $s = 0$ and r additive. (6) tells us that

$$\begin{pmatrix} r(\lambda\mu^{-1}x) \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} h_x \begin{pmatrix} 0 & \dfrac{\theta}{\lambda\bar{\mu}} \\ \dfrac{\bar{\theta}}{\bar{\lambda}\mu} & 0 \end{pmatrix}.$$

If $\lambda \in K_0$ and $\mu = 1$, this gives $r(\lambda x) = \sqrt{\lambda}\, r(x)$. Taking $\lambda = \eta = \sqrt{\theta}$ and $\mu = \bar{\eta}$, it says $r(\omega\theta x) = \eta c(x)$, so $\omega^2 r(\theta) = c(1)$.

Let $h_x(B) = \binom{u(x)}{v(x)}$. We have

$$\begin{pmatrix} u(x+y) \\ v(x+y) \end{pmatrix} = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} + \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h_y \left(\begin{pmatrix} x\bar{x} & x \\ \bar{x} & 1 \end{pmatrix} \right).$$

Take $x = y \in K_0$ to obtain $0 = \sqrt{x}\, c(y) + xv(y) = xc(1) + xv(x)$, whence $x(c(1) + v(x)) = 0$. We also see that v is additive, so $c(1) = 0$ and u is additive. It follows that $r(\theta) = \omega\eta c(1) = 0$, and we take $\lambda = \theta$, $\mu = \theta^{-1}$ above to observe that

$$\begin{pmatrix} r(\theta^2) \\ 0 \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix} h_1(\bar{D}) = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix} h_1(C + D) = \begin{pmatrix} \theta r(1) + c(1) \\ 0 \end{pmatrix} = \begin{pmatrix} \theta r(1) \\ 0 \end{pmatrix}.$$

Now, we have seen that r is additive, so $r(\theta^2) = r(\theta + \omega) = r(\theta) + r(\omega) = \theta r(1)$. We have just seen $r(\theta) = 0$ and we know $r(\omega) = \omega^2 r(1)$, so $(\omega^2 + \theta) r(1) = 0$. This implies $r(1) = 0$; together $r(1) = 0$ and $r(\theta) = 0$ imply $r \equiv 0$, and this in turn gives $c \equiv 0$.

u is additive, and it follows from (6) that $u(\lambda x) = \lambda u(x)$ for $\lambda \in K_0$. Thus, to determine u, we need to find $u(\theta)$. We get

$$u(\lambda\mu^{-1}x) = \frac{\lambda}{\sqrt{\mu\bar{\mu}}}u(x)$$

from (6). Taking $\lambda = \theta$, $\mu = \theta^{-1}$, we have $u(\theta^2) = \theta\sqrt{\theta\bar{\theta}}\,u(1) = \theta\omega^2 u(1)$. But $u(\theta^2) = u(\theta + \omega) = u(\theta) + u(\omega) = u(\theta) + \omega u(1)$. Then $u(\theta) = (\omega + \theta\omega^2)u(1)$.

Taking $\lambda = \eta$ and $\mu = \eta^{-1}$, we get directly $u(\theta) = \eta\omega u(1)$. So

$$(\omega + \theta\omega^2 + \eta\omega)u(1) = 0.$$

We have $(\omega + \theta\omega^2 + \eta\omega)^2 = \omega^2(1 + \omega^2\theta^2 + \theta) = \theta \neq 0$, so $u(1) = 0$. This implies $u \equiv 0$, and we have shown that f vanishes on a 2-Sylow subgroup of L. We conclude that $H^1(L, H^1(U, W) = 0$.

This completes the proof that $H^2(P, W) = 0$. Similar arguments give $H^2(P, V/W) = 0$, and together these results imply $H^2(P, V) = 0$.

4. $H^2(SL(3, 3^k), V)$ is non-zero if $k > 1$, but the cohomology group vanishes when $k = 1$ [6]. We saw above that $H^2(SU(3, 3^k), V)$ is also non-zero for $k > 1$. In the next theorem we show that the analogy is complete.

THEOREM 3. $H^2(SU(3, 3), V) = 0$.

Proof. Take B, U, Z and T as in Section 2. Let

$$0 \to V \to E \xrightarrow{\pi} SU(3, 3) \to 1$$

be an extension of $SU(3, 3)$ by V; we will show the extension is split.

Choose e and f in E with

$$\pi(e) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & -a^3 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \pi(f) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Computation gives the commutator

$$\left[\begin{pmatrix} w_1 & & v_1 \\ w_2 & e, & v_2 \\ w_3 & & v_3 \end{pmatrix} f\right] = \begin{pmatrix} av_2 + bv_3 - cw_3 \\ a^3 v_3 \\ 0 \end{pmatrix} [e, f]. \tag{7}$$

If the bottom component of $[e, f]$ is non-zero, there is a non-trivial $\mathbf{F}_3 T$-map

$$U/Z \otimes_{\mathbf{F}_3} Z \to V_3 = \left\{ \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \,\middle|\, v_3 \in K \right\},$$

but, checking the eigenvalues of the T-action on U/Z, Z and V_3, we see this is impossible. Hence the V_3 component of $[e, f]$ is zero.

Elementary arguments about the cohomology of cyclic groups show that we can choose e and f of order 3. Take x of order 3 with

$$\pi(x) = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

x is inverted by an involution in E; we may assume that involution is t^4, where

$$\pi(t) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^{-3} \end{pmatrix}$$

and λ is a generator of the multiplicative group of $\mathbf{F}_9$ satisfying $\lambda^2 + \lambda - 1 = 0$.

It is clear from (7) that we can choose a y_1 with

$$\pi(y_1) = \begin{pmatrix} 1 & 0 & \lambda^6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad [x, y_1] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In fact, x commutes with the nine elements of shape

$$\begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} y_1.$$

Computation shows that all the elements of the coset Vy_1 have order 3, and we see that we can choose an element y in this coset such that t^2 centralizes y and $[x, y]$ has shape

$$\begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix}.$$

If e as above has order 3 and $a \neq 0$, then, for $v \in V$, ve has order 3 if and only if

$$v = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}.$$

It follows from (7) that ve has order 3 if and only if $[ve, y] = [e, y]$. We will use this property to identify elements of order 3.

Let $z = t^2xt^{-2}$ and consider xz, which is in the same coset as $t^{-1}xty$. We claim $t^{-1}xty$ has order 3. Since $t^2yt^{-2} = y$, $tyt^{-1} = y^{-1}$. Thus $o(t^{-1}xty) = o(xy^{-1})$. $(xy^{-1})^3 = [y^{-1}, x]^x[x^{-1}, y]$, but x centralizes $[x, y]$ and thus $[y^{-1}, x]$, so this is

$$[y^{-1}, x][x^{-1}, y] = [y^{-1}, x][x, y]^{t^4} = [y^{-1}, x][x, y]^{-1} = [x, y]^{y^{-1}}[x, y]^{-1}.$$

But y^{-1} acts trivially on $[x, y]$ since $[x, y]$ has shape

$$\begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix},$$

and we have

$$(xy^{-1})^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, to see $o(xz) = 3$, it suffices to show $[xz, y] = [t^{-1}xty, y]$. Straightforward computation shows that

$$[t^{-1}xty, y] = -\lambda^7[x, y] \quad \text{and} \quad [xz, y] = (1 + \lambda^2)[x, y].$$

Since $\lambda^2 + \lambda = 1$, the computators are equal and xz had order 3.

Now take $a = t^4x$, $b = t^4z^{-1}$ and $c = t^4$. $\{a, b, c\}$ is a set of 3-transpositions no two of which commute. We have $b^c \neq (b^c)^a$, so, by a theorem of Fischer [3], $\langle a, b, c\rangle$ is a homomorphic image of a group of order 54. Since $\langle a, b, c\rangle$ covers $\langle \pi(t^4), U\rangle$, a subgroup of order 54 of $SU(3, 3)$, $\langle a, b, c\rangle$ has order 54. It follows that the extension splits.

McLaughlin and Griess have obtained other proofs of this result.

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